ERROR ANALYSIS FOR A FINITE ELEMENT APPROXIMATION OF ELLIPTIC DIRICHLET BOUNDARY CONTROL PROBLEMS

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Abstract. We consider the Galerkin finite element approximation of an elliptic Dirichlet boundary control model problem governed by the Laplacian operator. The functional theoretical setting of this problem uses L^2 controls and a "very weak" formulation of the state equation. However, the corresponding finite element approximation uses standard continuous trial and test functions. For this approximation, we derive a priori error estimates of optimal order, which are confirmed by numerical experiments. The proofs employ duality arguments and known results from the L^p error analysis for the finite element Dirichlet and Neumann projection.

Key words. Dirichlet boundary control, finite elements, a priori error estimates.

AMS subject classifications. 65K10, 65N30, 65N21, 49M25, 49K20.

1. Introduction and statement of results. We consider the following elliptic Dirichlet boundary control problem posed on a convex polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$:

$$\min\ J(u,q) := \tfrac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \tfrac{\alpha}{2}\|q\|^2_{L^2(\Gamma)} \tag{1.1}$$

with respect to $\{u,q\}$, under the constraint

$$-\Delta u = f \quad \text{in } \Omega, \quad u = q \ \text{ on } \Gamma. \tag{1.2}$$

Here, u_d and f are sufficiently smooth prescribed functions, while $\alpha > 0$ is a regularization parameter. For simplicity, we assume that at least $u_d, f \in L^2(\Omega)$. The natural functional analytic setting of this problem, which is also most convenient for numerical approximation, uses $Q := L^2(\Gamma)$ as "control space". This prohibits the choice of the associated "state space" to be $H^1(\Omega)$ as the trace operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is not surjective. To overcome this dilemma, we use a "very weak" formulation of the state equation (1.2) allowing for solutions $u \in L^2(\Omega)$ (see Grisvard [9], [10], and Berggren [3]): *For given* $q \in L^2(\Gamma)$ *find* $u \in L^2(\Omega)$ *such that*

$$-(u,\Delta\varphi) + \langle q, \partial_n\varphi\rangle = (f,\varphi) \quad \forall \varphi \in H^1_0(\Omega) \cap H^2(\Omega). \tag{1.3}$$

Here, $(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}$ is the L^2 inner product on the domain Ω and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L^2(\Gamma)}$ that on its boundary Γ. The corresponding norms are $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$ and $|\cdot| = |\cdot|_{L^2(\Gamma)}$, respectively. There are alternative variational formulations of Dirichlet boundary optimal control problems of the type (1.1), (1.2). For a brief survey, we refer to Kunisch/Vexler [12]. However, the formulation considered here appears to be the most attractive one from the computational point of view.

The finite element discretization of this optimization problem uses a standard weak formulation of the state equation which is possible due to higher regularity of

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the actual solution pair $\{\hat{u}, \hat{q}\}$. For this approximation the estimate

$$|\hat{q} - \hat{q}_h| + \|\hat{u} - \hat{u}_h\| = \mathcal{O}(h^{1-1/p}) \tag{1.4}$$

has been given in Casas/Raymond [5] for a problem with additional inequality constraints for the control q, which was expected to be only suboptimal for the state. The contribution of this paper consists in the improved L^2-error estimate

$$\|\hat{u} - \hat{u}_h\| = \mathcal{O}(h^{3/2-1/p}). \tag{1.5}$$

and in "optimal-order" error estimates with respect to weaker norms of the form

$$|\hat{q} - \hat{q}_h|_{\tilde{H}^{-1/2}(\Gamma)} = \mathcal{O}(h^{3/2-1/p-1/r}), \quad \|\hat{u} - \hat{u}_h\|_{H^{-1}(\Omega)} = \mathcal{O}(h^{2-1/p-1/r}). \tag{1.6}$$

Here, the values of $p \in [2, p_*^d)$ and $r \in [2, p_*^\Omega)$ depend on the regularity of the data and the limited elliptic regularity on domains with corners, respectively. Further, for the associated adjoint state the error estimate

$$\|\hat{z} - \hat{z}_h\| = \mathcal{O}(h^{2-1/p-1/r}) \tag{1.7}$$

is obtained. In view of the results of the computational experiments presented at the end of this paper, these estimates for the primal state and the control seem to be order-optimal. For maximum regularity, e.g., for smooth data on a rectangular domain, almost the order $\mathcal{O}(h^2)$ of convergence is obtained which is best possible for linear or bilinear finite elements. The proofs employ duality arguments based on the KKT system (**K**arush-**K**uhn-**T**ucker system) associated to the optimization problem (1.1), (1.2) and uses various results from the finite element error analysis in L^p for $p \neq 2$.

The contents of this paper is organized as follows. Section 2 contains the variational formulation of the Dirichlet boundary control problem and its Galerkin finite element approximation. This includes the derivation of the continuous and discrete KKT systems representing the first order necessary optimality conditions which form the basis of the error analysis. In Section 3 several auxiliary results on elliptic regularity and finite element approximation are provided. These are used in Section 4 to prove some suboptimal-order error estimates, followed by the final optimal-order ones in Section 5. The last Section 6 contains the results of some test calculations made to check the theoretical predictions.

2. The Dirichlet boundary control problem.

2.1. The state equation in "very weak" form. For later use, we provide some notation and results from the theory of Sobolev spaces and elliptic boundary value problems. The standard Sobolev spaces on Ω and Γ will be denoted by $H^s(\Omega)$, $H_0^s(\Omega)$, $W^{s,p}(\Omega)$, and $H^s(\Gamma)$, respectively, for $s \in \mathbb{R}_+$ and $1 \leq p \leq \infty$ (see Adams/Fournier [1]). For functions $v \in H^1(\Omega)$ the "strong" traces $v|_\Gamma \in L^2(\Gamma)$ exist and form the natural "trace space" $H^{1/2}(\Gamma)$ equipped with the generic norm

$$|v|_{H^{1/2}(\Gamma)} := \inf\left\{\|\psi\|_{H^1(\Omega)},\, \psi \in H^1(\Omega),\, \psi|_\Gamma = v\right\}.$$

Accordingly, we have the following "trace inequality" for functions $v \in H^1(\Omega)$:

$$|v|_{H^{1/2}(\Gamma)} \leq c\|v\|_{H^1(\Omega)}. \tag{2.1}$$

For right hand side $f \in L^2(\Omega)$ and boundary function $q \in H^{1/2}(\Gamma)$ the boundary value problem (1.2) has a standard "weak" solution $u \in H^1(\Omega)$, which is determined by $u|_\Gamma = q$ and

$$(\nabla u, \nabla\varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega). \tag{2.2}$$

For $q = 0$, this weak solution is in $H^2(\Omega)$ as Ω is a convex polygonal domain. Furthermore, it obeys the a priori bound

$$\|u\|_{H^2(\Omega)} \le c\|f\|. \tag{2.3}$$

Let $H^{-1/2}(\Gamma)$ denote the dual space of $H^{1/2}(\Gamma)$ equipped with the natural norm

$$|v|_{H^{-1/2}(\Gamma)} := \sup_{\chi \in H^{1/2}(\Gamma)} \frac{\langle v, \chi\rangle}{|\chi|_{H^{1/2}(\Gamma)}}.$$

For functions $v \in H^2(\Omega)$ the gradient ∇v has a trace $\nabla v|_\Gamma \in H^{1/2}(\Gamma)^2$. On a domain with smooth boundary Γ the outer normal unit vector n is continuous and, thus, the normal derivative $\partial_n v = n \cdot \nabla v \in H^{1/2}(\Gamma)$ is well defined for functions $v \in H^2(\Omega)$ and satisfies

$$|\partial_n v|_{H^{1/2}(\Gamma)} \le c\|v\|_{H^2(\Omega)}, \quad v \in H^2(\Omega). \tag{2.4}$$

This estimate does not make sense if Γ is polygonal, i.e. only Lipschitz continuous. But for $v \in H^2(\Omega)$, we still have $\partial_n v|_{\Gamma_i} \in H^{1/2}(\Gamma_i)$ on each of the straight components Γ_i, $i = 1, \dots, m$, of Γ. Accordingly, we introduce the space $\tilde{H}^{1/2}(\Gamma) := \{q \in L^2(\Gamma),\, q \in H^{1/2}(\Gamma_i),\, i = 1, \dots, m\}$. Further, on $L^2(\Gamma)$, we define the dual norm

$$|v|_{\tilde{H}^{-1/2}(\Gamma)} := \sup_{\psi \in H_0^1(\Omega) \cap H^2(\Omega)} \frac{\langle v, \partial_n\psi\rangle}{\|\psi\|_{H^2(\Omega)}} \le \sup_{\chi \in \tilde{H}^{1/2}(\Gamma)} \frac{\langle v, \chi\rangle}{|\chi|_{\tilde{H}^{1/2}(\Gamma)}}$$

and denote by $\tilde{H}^{-1/2}(\Gamma)$ the completion of $L^2(\Gamma)$ with respect to this norm. In the case of a smooth boundary Γ, the mapping $\partial_n : H_0^1(\Omega) \cap H^2(\Omega) \to H^{1/2}(\Gamma)$ is onto and we therefore have $\tilde{H}^{-1/2}(\Gamma) = H^{-1/2}(\Gamma)$.

The following lemma states the well-posedness of the general boundary value problem (1.3) in the "very weak" form. As a special case it also guarantees the existence of the "very weak" harmonic extension of general boundary data $q \in \tilde{H}^{-1/2}(\Gamma)$.

LEMMA 2.1. *For any given $q \in \tilde{H}^{-1/2}(\Gamma)$ the state equation in its "very weak" form (1.3) possesses a unique solution $u = u(q) \in L^2(\Omega)$. There holds the a priori estimate*

$$\|u\| \le c|q|_{\tilde{H}^{-1/2}(\Gamma)} + c\|f\|_{H^{-2}(\Omega)}, \tag{2.5}$$

where $H^{-2}(\Omega)$ denotes the dual space of $H_0^1(\Omega) \cap H^2(\Omega)$.

Proof. First, suppose that $q \in H^{1/2}(\Gamma)$ and $f \in H^{-1}(\Omega)$. Then, there exists a unique "weak" solution $u = u(q) \in H^1(\Omega)$ of the boundary value problem (2.2). By integration by parts, we find that this solution fulfills

$$-(u, \Delta\varphi) + \langle q, \partial_n\varphi\rangle = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \cap H^2(\Omega).$$

To prove the a priori estimate, we use a duality argument. Let $w \in H_0^1(\Omega)$ be the solution of the auxiliary problem

$$-\Delta w = u \ \text{ in } \Omega, \quad w|_\Gamma = 0.$$

By elliptic regularity, we have $w \in H^2(\Omega)$ and conclude

$$\|u\|^2 = (u, -\Delta w) = (f, w) - \langle q, \partial_n w \rangle.$$

Hence, using the dual norms defined above gives us

$$\|u\|^2 \le c\big\{\|f\|_{H^{-2}(\Omega)} + |q|_{\tilde{H}^{-1/2}(\Gamma)}\big\}\|w\|_{H^2(\Omega)}.$$

Since $\|w\|_{H^2(\Omega)} \le c\|u\|$, the a priori estimate (2.5) follows. Now, since the subspaces $H^{-1}(\Omega) \subset H^{-2}(\Omega)$ and $H^{1/2}(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma)$ are dense, the existence of a solution to the very weak variational problem for given data $f \in H^{-2}(\Omega)$ and $q \in \tilde{H}^{-1/2}(\Gamma)$ follows by a standard continuation argument. The a priori bound (2.5) carries over to these solutions by continuity and therefore also implies uniqueness. ◻

We will call a function $v \in L^2(\Omega)$ "very weakly harmonic" if it satisfies

$$(v, \Delta\varphi) - \langle q, \partial_n \varphi \rangle = 0 \quad \forall \varphi \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.6}$$

with some function $q \in \tilde{H}^{-1/2}(\Gamma)$. Then, the function q is the "very weak" trace of v on Γ. For this, almost by definition, we have the following trace estimate:

$$|q|_{\tilde{H}^{-1/2}(\Gamma)} = \sup_{\psi \in H_0^1(\Omega)\cap H^2(\Omega)} \frac{(v, \Delta\psi)}{\|\psi\|_{H^2(\Omega)}} \le c\|v\|. \tag{2.7}$$

We also call $Bq := v$ the "harmonic extension" of the boundary data $q \in \tilde{H}^{-1/2}(\Gamma)$ to Ω. For $q \in H^{1/2}(\Gamma)$, there holds $Bq \in H^1(\Omega)$ and

$$(\nabla Bq, \nabla\varphi) = 0 \quad \forall \varphi \in H_0^1(\Omega), \quad Bq|_\Gamma = q. \tag{2.8}$$

In the course of the further analysis, we will frequently use certain Sobolev trace inequalities and a priori bounds for the harmonic extension. From Grisvard [9, Theorem 1.5.1.2 f.] and the literature cited therein, we have the following result.

LEMMA 2.2. *Let $\Omega \in \mathbb{R}^d$ be a bounded domain with Lipschitz boundary Γ. For $1 < p < \infty$ and $s < 1 + 1/p$, such that $s - 1/p$ is not an integer, the trace operator is continuously defined from $W^{s,p}(\Omega)$ onto $W^{s-1/p,p}(\Gamma)$, with a right continuous inverse, and there holds*

$$|v|_{W^{s-1/p,p}(\Gamma)} \le c\|v\|_{W^{s,p}(\Omega)}. \tag{2.9}$$

Particularly, the harmonic extension is continuously defined from $W^{s-1/p,p}(\Gamma)$ into $W^{s,p}(\Omega)$ and satisfies

$$\|Bq\|_{W^{s,p}(\Omega)} \le c|q|_{W^{s-1/p,p}(\Gamma)}. \tag{2.10}$$

We note that in [9] the result of Lemma 2.2 is stated for $1/p < s < 1 + 1/p$, and only the existence of a linear extension operator satisfying (2.10) is claimed. Here, we assume that the estimate (2.9) holds for the slightly larger parameter range $0 < s < 1 + 1/p$ and that the estimate (2.10) actually holds for the *harmonic* extension.

The following lemmas collect some further properties of the trace operator and the harmonic extension which are not directly contained in Lemma 2.2.

LEMMA 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with boundary Γ.*

(i) For any fixed $p>2$ *, the trace operator is continuously defined from* $W^{1/2,p}(\Omega)$ *into* $L^p(\Gamma)$ *and there holds*

$$|v|_{L^p(\Gamma)} \le c\|v\|_{W^{1/2,p}(\Omega)}. \tag{2.11}$$

(ii) For $v \in H_0^1(\Omega)\cap W^{2,p}(\Omega)$ *,* $2<p<p_*^\Omega$*, with* p_*^Ω *defined in (2.32) below, the normal derivative* $\partial_n v$ *exists in* $W^{1-1/p,p}(\Gamma)$ *and there holds*

$$|\partial_n v|_{W^{1-1/p,p}(\Gamma)} \le c\|v\|_{W^{2,p}(\Omega)}. \tag{2.12}$$

In the exceptional case $p=2$*, the normal derivative* $\partial_n v$ *exists in* $H^s(\Gamma)$ *for* $0\le s<1/2$ *and there holds*

$$|\partial_n v|_{H^s(\Gamma)} \le c\|v\|_{H^2(\Omega)}. \tag{2.13}$$

(iii) For $q\in H^{1-1/p}(\Gamma)$ *,* $2\le p<\infty$ *, the normal derivative* $\partial_n Bq \in H^{-1/p}(\Gamma)$ *exists and there holds*

$$|\partial_n Bq|_{H^{-1/p}(\Gamma)} \le c|q|_{H^{1-1/p}(\Gamma)}. \tag{2.14}$$

In all cases the constants c *may depend on* p *or* s *, respectively.*

Proof. (i) For the estimate (2.11), we refer to Scott [17].
(ii) The estimate (2.12) can be derived by using the estimate (2.11) for the partial derivatives $\partial_i v$ and observing that $\partial_n v(x)\to 0$ as x approaches a corner point of Γ . For details see Jakovlev [11] where also the estimate (2.13) can be found.
(iii) For $q\in H^{1-1/p}(\Gamma)$ the results of Lemma 2.2, for $s=3/2-1/p$, imply that $Bq\in H^{3/2-1/p}(\Omega)$ and thus $\nabla Bq \in H^{1/2-1/p}(\Omega)^2$. Then, again using Lemma 2.2, we obtain $\partial_n Bq\in H^{-1/p}(\Gamma)$ and the asserted estimate (2.14). This completes the proof. □

Lemma 2.4. *For functions* $v\in W^{1,p}(\Omega)$ *,* $1<p<\infty$ *, there holds*

$$|v|_{L^p(\Gamma)} \le \varepsilon\|\nabla v\|_{L^p(\Omega)} + cp^{\frac{1}{p-1}}\varepsilon^{-\frac{1}{p-1}}\|v\|_{L^p(\Omega)}, \quad 0<\varepsilon\le 1. \tag{2.15}$$

Proof. For completeness, we supply the simple proof (see also Grisvard [9, Theorem 1.5.1.10]. Let $p'=p(p-1)^{-1}$. By the usual trace inequality for $W^{1,1}$ functions, there holds

$$|v|^p_{L^p(\Gamma)} \le c\|v^p\|_{L^1(\Omega)} + c\|\nabla v^p\|_{L^1(\Omega)}.$$

Then, using Hölder's and Young's inequality gives us

$$\begin{aligned}
|v|^p_{L^p(\Gamma)} &\le c\|v\|^p_{L^p(\Omega)} + cp\|\nabla v\|_{L^p(\Omega)}\|v^{p-1}\|_{L^{p'}(\Omega)}\\
&= c\|v\|^p_{L^p(\Omega)} + \big(\varepsilon\|\nabla v\|_{L^p(\Omega)}\big)\big(c\varepsilon^{-1}p\|v\|^{p-1}_{L^p(\Omega)}\big)\\
&\le c\|v\|^p_{L^p(\Omega)} + \varepsilon^p\|\nabla v\|^p_{L^p(\Omega)} + c^{p'}p^{p'}\varepsilon^{-p'}\|v\|^p_{L^p(\Omega)}.
\end{aligned}$$

This implies the asserted estimate. □

2.2. The optimization problem. For deriving existence results for the optimization problem considered, we define the "solution operator" $S\colon L^2(\Gamma) \to L^2(\Omega)$ by $Sq = u(q) = u$ and

$$-(u, \Delta\varphi) + \langle q, \partial_n \varphi \rangle = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \cap H^2(\Omega).$$

This operator is affine linear and continuous due to the a priori estimate (2.5). Then, the optimization problem can be rephrased in the following compact form:

$$j(q) := J(Sq, q) \to \min \quad \text{for } q \in L^2(\Gamma). \tag{2.16}$$

LEMMA 2.5. *The optimization problem (1.1) together with the very weak formulation (1.3) of the state equation possesses a uniquely determined solution* $\{\hat{u}, \hat{q}\} \in L^2(\Omega) \times L^2(\Gamma)$ *. This solution satisfies the necessary and in this case also the sufficient optimality condition*

$$\langle j'(\hat{q}), \chi \rangle = 0 \quad \forall \chi \in L^2(\Gamma), \tag{2.17}$$

with the derivative $j'(\hat{q}) : L^2(\Gamma) \to L^2(\Gamma)^* \simeq L^2(\Gamma)$ *.*

Proof. (i) The proof of existence is by the direct method of variational calculus. The reduced functional $j(\cdot)$ is bounded from below on $L^2(\Gamma)$ and strictly convex since the solution operator S is affine linear. Therefore, it is weakly lower semicontinuous in $L^2(\Gamma)$. Hence, there exists a minimizing sequence $(q_k)_{k\in\mathbb{N}}$, $\inf_{q\in L^2(\Gamma)} j(q) = \lim_{k\to\infty} j(q_k)$, which is bounded in $L^2(\Gamma)$ due to the coercivity of $j(\cdot)$ on $L^2(\Gamma)$. For any of its weak accumulation points $\hat{q}$ there holds $j(\hat{q}) \le \lim_{k\to\infty} j(q_k)$. Such an accumulation point is a unique global minimum of the reduced functional.
(ii) To prove the necessary optimality condition, we note that

$$\frac{1}{\varepsilon}\big(j(\hat{q} + \varepsilon\chi) - j(\hat{q})\big) \ge 0,$$

for any $\chi \in L^2(\Gamma)$. Then, letting $\varepsilon \to 0$ yields $\langle j'(\hat{q}), \chi \rangle \ge 0$ for all $\chi \in L^2(\Gamma)$, which implies (2.17). As the reduced cost functional is strictly convex, the condition (2.17) is not only necessary but also sufficient. □

LEMMA 2.6. *The directional derivative of* $j(\cdot)$ *at some point* $q \in L^2(\Gamma)$ *is given by*

$$j'(q)(\chi) = \alpha\langle q, \chi \rangle - \langle \partial_n z, \chi \rangle \tag{2.18}$$

for $\chi \in L^2(\Gamma)$ *, where* $z = z(q) \in H_0^1(\Omega) \cap H^2(\Omega)$ *is the solution of the associated "adjoint problem"*

$$-(\psi, \Delta z) = (Sq - u_d, \psi) \quad \forall \psi \in L^2(\Omega). \tag{2.19}$$

Proof. We introduce the Lagrangian functional $\mathcal{L}\colon L^2(\Omega) \times L^2(\Gamma) \times \{H_0^1(\Omega) \cap H^2(\Omega)\}$
$\to \mathbb{R}$ by $\mathcal{L}(u, q, z) := J(u, q) + (f, z) + (u, \Delta z) - \langle q, \partial_n z \rangle$. Then, with $u(q) = Sq$, there holds $j(q) = J(u(q), q) = \mathcal{L}(u(q), q, z(q))$, and for $\chi \in L^2(\Gamma)$:

$$\begin{aligned} j'(q)(\chi) = {}& \mathcal{L}'_u(u(q), q, z(q))(u'_q(q)(\chi)) + \mathcal{L}'_q(u(q), q, z(q))(\chi) \\ & + \mathcal{L}'_z(u(q), q, z(q))(z'_q(q)(\chi)). \end{aligned}$$

Since by construction $\mathcal{L}'_u(u(q),q,z(q))(\cdot)=0$ and $\mathcal{L}'_z(u(q),q,z(q))(\cdot)=0$, we obtain

$$j'(q)(\chi)=\mathcal{L}'_q(u(q),q,z(q))(\chi)=\alpha\langle\chi,q\rangle-\langle\chi,\partial_n z\rangle,$$

which proves the asserted representation. □

As a consequence of the foregoing lemmas, we have the following result.

LEMMA 2.7. *The solution of the optimization problem (1.1), (1.3) is characterized by the Euler-Lagrange principle stating that the pair* $\{\hat u,\hat q\}\in L^2(\Omega)\times L^2(\Gamma)$ *is a solution if and only if there exists an "adjoint state"* $\hat z\in H^1_0(\Omega)\cap H^2(\Omega)$, *such that the triplet* $\{\hat u,\hat q,\hat z\}$ *solves the "optimality system" (**K**arush-**K**uhn-**T**ucker system)*

$$-(\hat u,\Delta\varphi)+\langle\hat q,\partial_n\varphi\rangle=(f,\varphi)\qquad\forall\varphi\in H^1_0(\Omega)\cap H^2(\Omega),\tag{2.20}$$

$$\alpha\langle\hat q,\chi\rangle-\langle\partial_n\hat z,\chi\rangle=0\qquad\forall\chi\in L^2(\Gamma),\tag{2.21}$$

$$-(\psi,\Delta\hat z)-(\hat u,\psi)=-(u_d,\psi)\quad\forall\psi\in L^2(\Omega).\tag{2.22}$$

Proof. Let $\{\hat u,\hat q\}\in L^2(\Omega)\times L^2(\Gamma)$ be a solution of the optimization problem. Then, by definition there hold (2.20) and (2.22). The necessary condition (2.17) and the representation (2.18) of $j'(\cdot)$ imply (2.21). In turn, for each solution $\{\hat u,\hat q,\hat z\}\in L^2(\Omega)\times L^2(\Gamma)\times\{H^1_0(\Omega)\cap H^2(\Omega)\}$ of the KKT system, the necessary (and sufficient) optimality condition (2.17) is satisfied implying that $\hat q$ is a minimum. □

2.3. Galerkin finite element approximation. For the approximation of the optimization problem (1.1), (1.3), we consider a finite element method. Let $V_h\subset H^1(\Omega)$ be finite element subspaces defined on meshes $\mathbb{T}_h=\{K\}$ consisting of triangles or quadrilaterals and satisfying the usual conditions of shape and size regularity. These meshes are characterized by the mesh-width parameter $h\in(0,1]$ where $h\approx\max\{\mathrm{diam}(K),\,K\in\mathbb{T}_h\}$. Further, we set $V_{h,0}:=V_h\cap H^1_0(\Omega)$ and let V_h^∂ be the trace space corresponding to V_h. For simplicity, we only consider lowest-order finite elements, i.e., piecewise linear or bilinear trial and test functions. The discrete solutions $\{\hat u_h,\hat q_h\}\in V_h\times V_h^\partial$ are determined by the discrete optimization problems

$$J(\hat u_h,\hat q_h)=\min_{u_h\in V_h,q_h\in V_h^\partial}J(u_h,q_h),\tag{2.23}$$

under the constraints

$$(\nabla u_h,\nabla\varphi_h)=(f,\varphi_h)\quad\forall\varphi_h\in V_{h,0},\quad u_h|_\Gamma=q_h\,.\tag{2.24}$$

By analogous arguments as used for the continuous optimization problem (1.1), (1.3), we see that their discrete counterparts (2.23), (2.24) possess uniquely determined solutions. These solutions may be computed by applying the gradient or the Newton method for the discrete reduced functional $j_h(q_h):=J(S_hq_h,q_h)$, which requires the evaluation of first and second directional derivatives of $j_h(\cdot)$. Here, $S_h:V_h^\partial\to V_h$ denotes the discrete solution operator defined by $S_hq_h=u_h(q_h)=u_h$ and (2.24). This equation may be rewritten in the form

$$(\nabla v_h,\nabla\varphi_h)+(\nabla B_hq_h,\nabla\varphi_h)=(f,\varphi_h)\quad\forall\varphi_h\in V_{h,0},\tag{2.25}$$

for the function $v_h:=u_h-B_hq_h\in V_{h,0}$, where $B_h\colon V_h^\partial\to V_h$ is an arbitrary extension operator of discrete boundary data to all of $\bar\Omega$.

LEMMA 2.8. *With the foregoing notation the first directional derivative of* $j_h(\cdot)$ *at some point* $q_h \in V_h^\partial$ *is given by*

$$j_h'(q_h)(\chi_h) = \alpha\langle \chi_h, q_h\rangle + (B_h\chi_h, u_h - u_d) - (\nabla B_h\chi_h, \nabla z_h), \tag{2.26}$$

for $\chi_h \in V_h^\partial$, *where* $u_h = S_h q_h$ *and* $z_h = z_h(q_h) \in V_{h,0}$ *is the solution of the "discrete adjoint problem"*

$$(\nabla\psi_h, \nabla z_h) = (u_h - u_d, \psi_h) \quad \forall \psi_h \in V_{h,0}. \tag{2.27}$$

Proof. The argument is analogous to that used in Lemma 2.6 on the continuous level. For the discrete Lagrangian functional $\mathcal{L}_h \colon V_{h,0} \times V_h^\partial \times V_{h,0} \to \mathbb{R}$, defined by

$$\mathcal{L}_h(v_h, q_h, z_h) := J(v_h + B_h q_h, q_h) + (f, z_h) - (\nabla v_h, \nabla z_h) - (\nabla B_h q_h, \nabla z_h),$$

we have $j_h(q_h) = J(v_h + B_h q_h, q_h) = \mathcal{L}_h(v_h, q_h, z_h(q_h))$. Further,

$$\begin{aligned} j_h'(q_h)(\chi_h) &= \mathcal{L}_{h,v}'(v_h, q_h, z_h(q_h))(v_{h,q}'(\chi_h)) + \mathcal{L}_{h,q}'(v_h, q_h, z_h(q_h))(\chi_h) \\ &\quad + \mathcal{L}_{h,z}'(v_h, q_h, z_h(q_h))(z_{h,q}'(q_h)(\chi_h)) \\ &= \mathcal{L}_{h,q}'(v_h, q_h, z_h(q_h))(\chi_h) \\ &= (v_h + B_h q_h - u_d, B_h\chi_h) + \alpha\langle \chi_h, q_h\rangle - (\nabla B_h\chi_h, \nabla z_h) \\ &= (u_h - u_d, B_h\chi_h) + \alpha\langle \chi_h, q_h\rangle - (\nabla B_h\chi_h, \nabla z_h), \end{aligned}$$

which proves the asserted representation. □

As a consequence of the foregoing lemma, we obtain the following result which is analogous to the corresponding one on the continuous level, Lemma 2.7:

LEMMA 2.9. *The solution of the discrete optimization problem (2.23), (2.24) is characterized by the Euler-Lagrange principle stating that the pair* $\{\hat u_h, \hat q_h\} \in V_h \times V_h^\partial$ *is a solution if and only if there exists an "adjoint state"* $\hat z_h \in V_{h,0}$, *such that the triplet* $\{\hat u_h, \hat q_h, \hat z_h\}$ *solves the discrete KKT system*

$$\hat u_h|_\Gamma = \hat q_h, \quad (\nabla \hat u_h, \nabla\varphi_h) = (f, \varphi_h) \qquad \forall \varphi_h \in V_{h,0}, \tag{2.28}$$

$$\alpha\langle \hat q_h, \chi_h\rangle + (\hat u_h, B_h\chi_h) - (\nabla B_h\chi_h, \nabla \hat z_h) = (u_d, B_h\chi_h) \quad \forall \chi_h \in V_h^\partial, \tag{2.29}$$

$$(\nabla\psi_h, \nabla \hat z_h) - (\hat u_h, \psi_h) = -(u_d, \psi_h) \qquad \forall \psi_h \in V_{h,0}. \tag{2.30}$$

The solution $\{\hat u_h, \hat q_h, \hat z_h\}$ *is independent of the particular choice of the extension operator* B_h.

The numerical results for this approximation presented in Section 6 suggest the following partially "optimal" rates of convergence under generic assumptions on the regularity of the solution:

$$h^{1-1/r}|\hat q - \hat q_h| + h^{1/2-1/r}\|\hat u - \hat u_h\| + \|\hat z - \hat z_h\| = \mathcal{O}(h^{2-1/p-1/r}). \tag{2.31}$$

Here, $2 \le r < p_*^\Omega \le \infty$ and $2 \le p < p_*^d \le p_*$, $p_* := \min\{p_*^d, p_*^\Omega\}$ are essentially determined by the regularity $\hat z \in H_0^1(\Omega) \cap W^{2,p_*}(\Omega)$, depending on the maximum interior angle $\omega_{\max}$ of the polygonal domain Ω like

$$p_*^\Omega = \frac{2\omega_{\max}}{2\omega_{\max} - \pi}, \tag{2.32}$$

including the special case $p_*^\Omega = \infty$ for $\omega = \frac{\pi}{2}$, and the regularity of the data $u_d \in L^{p_*^d}(\Omega)$, $p_*^d > 2$. These convergence rates turn out to be better for weaker error measures, e.g., for the mean values:

$$|\langle \hat{q} - \hat{q}_h, 1\rangle| + |(\hat{u} - \hat{u}_h, 1)| = \mathcal{O}(h^{2-1/p-1/r}). \tag{2.33}$$

It is the main goal of the following analysis to provide theoretical support for these practically observed convergence rates. This will also cover the case of solutions with reduced regularity induced by irregular data.

REMARK 1. The assumption that the domain Ω is polygonal and convex is made for simplifying the arguments in the proofs. A curved boundary complicates the numerical approximation while "reentrant corners" reduce the solution's regularity. On a general polygonal (or piecewise smoothly bounded) domain, the optimization problem is well-posed if in the state equation (1.3) the test space $D(\Delta) := \{\varphi \in H_0^1(\Omega), \Delta\varphi \in L^2(\Omega)\}$ is used.

2.4. The KKT systems of the optimization problems. The error analysis of the finite element approximation (2.23), (2.24) of the optimization problem (1.1), (1.3) is based on their equivalent formulation in terms of the corresponding KKT systems. We begin by recasting the KKT system (2.20), (2.21), (2.22) in a form which can be approximated by a standard finite element method using only continuous trial and test functions. This is possible, since the solution of the very weak optimality system (2.20), (2.21), (2.22) turns out to be more regular than required for its definition. For later use, we determine its degree of regularity, which is guaranteed in general on a convex polygonal domain. Actually, the regularity of the solution pair $\{\hat{u}, \hat{q}\}$ is essentially determined by that of the adjoint state $\hat{z}$.

LEMMA 2.10. *Suppose that $f \in L^2(\Omega)$ and $u_d \in L^{p_*^d}(\Omega)$, $p_*^d > 2$. Let $p_*^\Omega \geq 2$ be defined by (2.32) and $p_* := \min\{p_*^d, p_*^\Omega\}$. Then, the solution $\{\hat{u}, \hat{q}\} \in L^2(\Omega) \times L^2(\Gamma)$ of the optimization problem (1.1), (1.3) and the associated adjoint state $\hat{z} \in H_0^1(\Omega) \cap H^2(\Omega)$ determined by (2.19) have the additional regularity properties*

$$\{\hat{u}, \hat{q}\} \in H^{3/2-1/p}(\Omega) \times H^{1-1/p}(\Gamma), \quad \hat{z} \in W^{2,p}(\Omega), \quad 2 \leq p < p_*. \tag{2.34}$$

Proof. By Lemma 2.7, the triplet $\{\hat{u}, \hat{q}, \hat{z}\}$ satisfies the equations (2.20), (2.21), (2.22). Since $\hat{q} \in L^2(\Gamma)$, according to Berggren [3], we have $\hat{u} \in H^s(\Omega)$ for $0 \leq s < \frac{1}{2}$ and therefore $\hat{u} \in L^p(\Omega)$ for some $p > 2$. Let this p be chosen such that $p < p_*$. This in turn implies that $\hat{z} \in W^{2,p}(\Omega)$ for this $p > 2$ (see Grisvard [9]). Hence, $\partial_n \hat{z} \in W^{1-1/p,p}(\Gamma)$ by Lemma 2.3(ii). Then, from (2.21), we infer that $\hat{q} \in W^{1-1/p,p}(\Gamma) \subset H^{1-1/p}(\Gamma) \subset H^{1/2}(\Gamma)$. Using this in (2.20) yields $\hat{u} \in H^1(\Omega)$, i.e., $\hat{u}$ is the usual "weak" H^1 solution of the boundary value problem (1.2). By elliptic regularity theory, this implies $\hat{z} \in W^{2,p}(\Omega)$ for $2 \leq p < p_*$. Finally, in view of Lemma 2.2, by elliptic regularity theory, $\hat{u}|_\Gamma = \hat{q} \in W^{1-1/p,p}(\Gamma) \subset H^{1-1/p}(\Gamma)$ implies that $\hat{u} \in H^{3/2-1/p}(\Omega)$, which completes the proof. ◻

REMARK 2. The right hand side in the equation for $\hat{z}$ is $\hat{u} - u_d$. Hence, under the mere assumption that $u_d \in L^\infty(\Omega)$, in general, we cannot expect $\hat{z} \in W^{2,\infty}(\Omega)$ or higher regularity, even on a rectangle. This restricts all our results to the case $2 \leq p < \infty$ with constants blowing up as $p \to \infty$. However, in the special case $\omega_{\max} = \frac{\pi}{2}$, we have (2.34) with $p = \infty$, provided that $u_d \in C^\gamma(\bar{\Omega})$ for some $\gamma > 0$ and $u_d(x_i) = 0$ in all cornerpoints x_i of Ω. This follows from Grisvard [9] due to the fact that $\hat{u}(x_i) = \hat{q}(x_i) = \alpha^{-1}\partial_n \hat{z}(x_i) = 0$, cf. the argument in the proof of Lemma 2.3 (ii), and $\hat{u} \in C^\gamma(\bar{\Omega})$ by an embedding theorem.

Next, we rewrite equation (2.21) using equation (2.22) to obtain

$$\begin{aligned}\alpha\langle \hat{q}, \chi\rangle - \langle \partial_n \hat{z}, \chi\rangle &= \alpha\langle \hat{q}, \chi\rangle - (\Delta \hat{z}, B\chi) - (\nabla \hat{z}, \nabla B\chi)\\ &= \alpha\langle \hat{q}, \chi\rangle + (\hat{u} - u_d, B\chi),\end{aligned}$$

where $B\chi \in H^1(\Omega)$ is the harmonic extension of the boundary function $\chi \in H^{1/2}(\Gamma)$ defined by (2.8). Hence, in view of Lemma 2.10, the solution $\{\hat{u}, \hat{q}, \hat{z}\} \in H^{3/2-1/p}(\Omega)\times W^{1-1/p,p}(\Gamma) \times [H_0^1(\Omega) \cap W^{2,p}(\Omega)]$ of (2.20), (2.21), (2.22) also satisfies the following set of equations:

$$\hat{u}|_\Gamma = \hat{q}, \quad (\nabla \hat{u}, \nabla \varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega), \tag{2.35}$$

$$\alpha\langle \hat{q}, \chi\rangle + (\hat{u}, B\chi) = (u_d, B\chi) \quad \forall \chi \in H^{1/2}(\Gamma), \tag{2.36}$$

$$(\nabla \psi, \nabla \hat{z}) - (\hat{u}, \psi) = -(u_d, \psi) \quad \forall \psi \in H_0^1(\Omega). \tag{2.37}$$

In order to remove the nonhomogeneous boundary condition, we introduce the function $\hat{v} := \hat{u} - B\hat{q} \in H_0^1(\Omega)$. Then, the triplet $\{\hat{v}, \hat{q}, \hat{z}\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$ satisfies the system

$$(\nabla \hat{v}, \nabla \varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega), \tag{2.38}$$

$$\alpha\langle \hat{q}, \chi\rangle + (\hat{v} + B\hat{q}, B\chi) = (u_d, B\chi) \quad \forall \chi \in H^{1/2}(\Gamma), \tag{2.39}$$

$$(\nabla \psi, \nabla \hat{z}) - (\hat{v} + B\hat{q}, \psi) = -(u_d, \psi) \quad \forall \psi \in H_0^1(\Omega). \tag{2.40}$$

The corresponding finite element approximation $\{\hat{u}_h, \hat{q}_h, \hat{z}_h\} \in V_h \times V_h^\partial \times V_{h,0}$ is characterized by the discrete KKT system (see Lemma 2.7)

$$\hat{u}_h|_\Gamma = \hat{q}_h, \quad (\nabla \hat{u}_h, \nabla \varphi_h) = (f, \varphi_h) \qquad \forall \varphi_h \in V_{h,0}, \tag{2.41}$$

$$\alpha\langle \hat{q}_h, \chi_h\rangle + (\hat{u}_h, B_h\chi_h) - (\nabla \hat{z}_h, \nabla B_h\chi_h) = (u_d, B_h\chi_h) \quad \forall \chi_h \in V_h^\partial, \tag{2.42}$$

$$(\nabla \psi_h, \nabla \hat{z}_h) - (\hat{u}_h, \psi_h) = -(u_d, \psi_h) \quad \forall \psi_h \in V_{h,0}, \tag{2.43}$$

or incorporating the nonhomogeneous boundary condition for $\hat{u}_h$ into the variational formulation:

$$(\nabla \hat{v}_h, \nabla \varphi_h) + (\nabla B_h\hat{q}_h, \nabla \varphi_h) = (f, \varphi_h) \qquad \forall \varphi_h \in V_{h,0}, \tag{2.44}$$

$$\alpha\langle \hat{q}_h, \chi_h\rangle + (\hat{v}_h + B_h\hat{q}_h, B_h\chi_h) - (\nabla \hat{z}_h, \nabla B_h\chi_h) = (u_d, B_h\chi_h) \quad \forall \chi_h \in V_h^\partial, \tag{2.45}$$

$$(\nabla \psi_h, \nabla \hat{z}_h) - (\hat{v}_h + B_h\hat{q}_h, \psi_h) = -(u_d, \psi_h) \quad \forall \psi_h \in V_{h,0}, \tag{2.46}$$

where $\hat{v}_h := \hat{u}_h - B_h\hat{q}_h \in V_{h,0}$. The solution of this system is independent of the particular choice of the extension operator $B_h : V_h^\partial \to V_h$.

From now on, we choose B_h to be the "discrete harmonic extension" defined by

$$(\nabla B_h q_h, \nabla \varphi_h) = 0 \quad \forall \varphi_h \in V_{h,0}, \quad B_h q_h|_\Gamma = q_h. \tag{2.47}$$

Then, the system (2.44), (2.45), (2.46) reduces to

$$(\nabla \hat{v}_h, \nabla \varphi_h) = (f, \varphi_h) \qquad \forall \varphi_h \in V_{h,0}, \tag{2.48}$$

$$\alpha\langle \hat{q}_h, \chi_h\rangle + (\hat{v}_h + B_h\hat{q}_h, B_h\chi_h) = (u_d, B_h\chi_h) \quad \forall \chi_h \in V_h^\partial, \tag{2.49}$$

$$(\nabla \psi_h, \nabla \hat{z}_h) - (\hat{v}_h + B_h\hat{q}_h, \psi_h) = (u_d, \psi_h) \qquad \forall \psi_h \in V_{h,0}. \tag{2.50}$$

Writing the equations for $\{\hat{v}, \hat{q}, \hat{z}\}$ for discrete test functions and subtracting the corresponding discrete equations (2.48), (2.49), (2.50) yields the following equations for the errors $e_v := \hat{v} - \hat{v}_h$, $e_q := \hat{q} - \hat{q}_h$, and $e_z := \hat{z} - \hat{z}_h$:

$$(\nabla e_v, \nabla \varphi_h) = 0 \quad \forall \varphi_h \in V_{h,0}, \quad (2.51)$$

$$\alpha \langle e_q, \chi_h \rangle + (\hat{v} + B\hat{q} - u_d, B\chi_h) - (\hat{v}_h + B_h\hat{q}_h - u_d, B_h\chi_h) = 0 \quad \forall \chi_h \in V_h^\partial, \quad (2.52)$$

$$(\nabla e_z, \nabla \psi_h) - (e_v + B\hat{q} - B_h\hat{q}_h, \psi_h) = 0 \quad \forall \psi_h \in V_{h,0}. \quad (2.53)$$

REMARK 3. Notice that, since $B_h \neq B$, the system (2.48), (2.49), (2.50) is *not* the Galerkin approximation of (2.38), (2.39), (2.40). In this situation the general paradigm that "Galerkin discretization" and "optimization" (i.e. forming the necessary optimality condition) commute does not hold. This essentially complicates the error analysis as several additional terms need to estimated, which originate from the lacking Galerkin orthogonality of the approximation.

REMARK 4. The choice of $B\colon H^{1/2}(\Gamma) \to H^1(\Omega)$ and $B_h : V_h^\partial \to V_h$ as the harmonic, respectively, discrete harmonic extension operators is for convenience of the argument used in the following error analysis. Actually, the continuous as well as the discrete optimal solutions are independent of this particular choice. For practical computations B_h is usually chosen to satisfy $B_h q_h(a_i) = 0$ in all interior nodal points a_i.

3. Auxiliary estimates.

3.1. Auxiliary error and stability estimates. Next, we collect some known results on the approximation behavior of finite element methods.

(I) We will use the following "inverse estimate" for finite element functions $\chi_h \in V_h^\partial$ (notice that Γ is one-dimensional):

$$|\chi_h|_{H^r(\Gamma)} \le ch^{s-r}|\chi_h|_{H^s(\Gamma)}, \quad (3.1)$$

for $0 \le s \le r \le 1$. This can be proven by combining estimates in Ciarlet [7] and Brenner/Scott [4] with standard results from operator interpolation theory.

(II) Let $I_h : C(\bar{\Omega}) \to V_h$ and $I_h : C(\Gamma) \to V_h^\partial$ denote the natural nodal interpolation operators which satisfy (see Ciarlet [7] and Brenner/Scott [4])

$$\|\nabla(v - I_h v)\|_{L^p(\Omega)} \le ch\|\nabla^2 v\|_{L^p(\Omega)}, \quad (3.2)$$

for $v \in W^{2,p}(\Omega)$, $1 \le p \le \infty$. The interpolation operator I_h is local, but only defined for continuous functions. A locally defined quasi-interpolation operator $\tilde{I}_h : L^2(\Omega) \to V_h$ has been devised by Scott/Zhang [18]. This operator preserves (polynomial) boundary conditions and satisfies $\tilde{I}_h|_{V_h} = id$. For this, there holds the same estimate (3.2) as for the standard nodal interpolation I_h and in addition the estimates

$$\|v - \tilde{I}_h v\| + h\|\nabla(v - \tilde{I}_h v)\| \le ch^s\|v\|_{H^s(\Omega)}, \quad 1 \le s \le 2, \quad (3.3)$$

for $v \in H^s(\Omega)$.

(III) The L^2 projection $P_h^\partial : L^2(\Gamma) \to V_h^\partial$ is defined by

$$\langle q - P_h^\partial q, \chi_h \rangle = 0 \quad \forall \chi_h \in V_h^\partial.$$

By standard results for finite elements there holds the error estimate (see Ciarlet [7], Brenner/Scott [4], and Casas/Raymond [6])

$$|q - P_h^\partial q|_{L^p(\Gamma)} + h^s |P_h^\partial q|_{W^{s,p}(\Gamma)} \le c h^s |q|_{W^{s,p}(\Gamma)}, \quad 0 \le s \le 1,\ 1 < p < \infty, \tag{3.4}$$

for $q \in W^{s,p}(\Gamma)$.

(IV) For a function $u \in H_0^1(\Omega)$ let $R_h^D u \in V_{h,0}$ denote the corresponding "Ritz projection" ("Dirichlet projection") defined by

$$(\nabla(u - R_h^D u), \nabla\varphi_h) = 0 \quad \forall \varphi_h \in V_{h,0}.$$

We recall the following standard results from the literature

Lemma 3.1. *On a convex polygonal domain, the "Ritz projection" $R_h^D : H_0^1(\Omega) \to V_{h,0}$ satisfies the stability estimate*

$$\|\nabla R_h^D u\|_{L^p(\Omega)} \le c \|\nabla u\|_{L^p(\Omega)}, \quad 1 < p \le \infty, \tag{3.5}$$

for $u \in W^{1,p}(\Omega)$. Furthermore, there holds the error estimate

$$\|u - R_h^D u\|_{L^p(\Omega)} + h\|u - R_h^D u\|_{W^{1,p}(\Omega)} \le c h^2 \|u\|_{W^{2,p}(\Omega)}, \quad 2 \le p < p_*^\Omega. \tag{3.6}$$

Proof. The proofs can be extracted from the results in Rannacher/Scott [15]. □

(V) Finally, we introduce the "Neumann projection" $R_h^N u \in V_h$ of a function $u \in H^1(\Omega)$ defined by

$$(\nabla(u - R_h^N u), \nabla\varphi_h) + (u - R_h^N u, \varphi_h) = 0 \quad \forall \varphi_h \in V_h.$$

For this approximation, we recall the following results from the literature.

Lemma 3.2. *On a convex polygonal domain, the "Neumann projection" R_h^N satisfies the error estimate*

$$\|u - R_h^N u\|_{L^p(\Omega)} + h^{1/p} |u - R_h^N u|_{L^p(\Gamma)} \le c h^2 \|u\|_{W^{2,p}(\Omega)}, \tag{3.7}$$

for $q_*^\Omega < p < p_*^\Omega$, where $q_*^\Omega = p_*^\Omega (p_*^\Omega - 1)^{-1}$.

Proof. The bound for the first term in (3.7) can be obtained by combining (in a nontrivial way) results from Rannacher/Scott [15] and Scott [16], which hold for $q_*^\Omega < p < p_*^\Omega$. One of these results is the estimate $\|\nabla(u - R_h^N u)\|_{L^p(\Omega)} \le c h \|u\|_{W^{2,p}(\Omega)}$. Then, the bound for the second term in (3.7) follows by using the relation (2.15) with $\varepsilon := h^{1-1/p}$,

$$\begin{aligned} |u - R_h^N u|_{L^p(\Gamma)} &\le c\varepsilon \|\nabla(u - R_h^N u)\|_{L^p(\Omega)} + c\varepsilon^{-1/(p-1)} \|u - R_h^N u\|_{L^p(\Omega)} \\ &\le c h^{1-1/p} \|\nabla(u - R_h^N u)\|_{L^p(\Omega)} + c h^{-1/p} \|u - R_h^N u\|_{L^p(\Omega)} \\ &\le c h^{2-1/p} \|u\|_{W^{2,p}(\Omega)}. \end{aligned}$$

This completes the proof. □

3.2. Properties of discrete harmonic extension. In this section, we provide some a priori bounds and error estimates for the discrete harmonic extensions.

LEMMA 3.3. *For the discrete harmonic extension* $B_hq_h \in V_h$ *of the boundary data* $q_h \in V_h^\partial$ *there hold the a priori estimates*

$$\|\nabla B_hq_h\| \le c|q_h|_{H^{1/2}(\Gamma)}, \tag{3.8}$$

$$\|B_hq_h\| \le c|q_h|, \tag{3.9}$$

$$\|\nabla B_hq_h\|_{L^p(\Omega)} \le ch^{1/p-1}|q_h|_{L^p(\Gamma)}, \quad 1 < p < \infty. \tag{3.10}$$

Proof. (i) The poof is by referring back to the corresponding estimates for B. For the modified interpolation operator $\tilde I_h$ in (3.3) there holds $(B_h - \tilde I_h B)q_h|_\Gamma = 0$. Hence by the properties of B_h and the estimate (3.3) for $s = 1$, we have

$$\begin{aligned}\|\nabla B_hq_h\|^2 &= (\nabla B_hq_h, \nabla(B_h - \tilde I_hB)q_h) + (\nabla B_hq_h, \nabla \tilde I_hBq_h)\\ &\le \|\nabla B_hq_h\|\|\nabla \tilde I_hBq_h\| \le c\|\nabla B_hq_h\|\|Bq_h\|_{H^1(\Omega)}.\end{aligned}$$

Then, the stability estimate (2.10) yields (3.8),

$$\|\nabla B_hq_h\| \le c\|Bq_h\|_{H^1(\Omega)} \le c|q_h|_{H^{1/2}(\Gamma)}.$$

(ii) Next, let $w \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of the auxiliary problem

$$-\Delta w = B_hq_h \quad \text{in } \Omega, \quad w|_\Gamma = 0.$$

Then, using several of the foregoing estimates in a standard way, we obtain

$$\begin{aligned}\|B_hq_h\|^2 &= (B_hq_h, -\Delta w) = (\nabla B_hq_h, \nabla w) - \langle q_h, \partial_n w\rangle\\ &= (\nabla B_hq_h, \nabla(w - R_h^D w)) - \langle q_h, \partial_n w\rangle\\ &\le \|\nabla B_hq_h\|\|\nabla(w - R_h^D w)\| + |q_h|\|w\|_{H^2(\Omega)}\\ &\le c\big\{h|q_h|_{H^{1/2}(\Gamma)} + |q_h|\big\}\|w\|_{H^2(\Omega)}\\ &\le c|q_h|\|B_hq_h\|.\end{aligned}$$

This yields (3.9).
(iii) To prove (3.10), we recall the estimate

$$\|\nabla v\|_{L^p(\Omega)} \le c \sup_{w \in W_0^{1,q}(\Omega)} \frac{(\nabla v, \nabla w)}{\|\nabla w\|_{L^q(\Omega)}}, \tag{3.11}$$

which holds for $1 < p < \infty$ and $q = p(p-1)^{-1}$, particularly on convex polygonal domains. This follows from a result in Alkhutov/Kondratev [2], which states that the boundary value problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u|_\Gamma = 0,$$

possesses for $f \in W^{-1,p}(\Omega)$ a uniquely determined solution $u \in W_0^{1,p}(\Omega)$ satisfying

$$\|u\|_{W^{1,p}(\Omega)} \le c\|f\|_{W^{-1,p}(\Omega)}.$$

For given $q_h \in V_h^\partial$ let $\tilde{B}_h q_h \in V_h$ denote that extension which coincides with q_h at each nodal point $a_i \in \Gamma$, but vanishes at each interior nodal point $a_i \in \Omega$. Then, for $v_h := B_h q_h - \tilde{B}_h q_h \in V_{h,0}$ there holds

$$(\nabla v_h, \nabla \varphi_h) = (\nabla B_h q_h, \nabla \varphi_h) - (\nabla \tilde{B}_h q_h, \nabla \varphi_h) = -(\nabla \tilde{B}_h q_h, \nabla \varphi_h), \quad \varphi_h \in V_{h,0}.$$

Now, let $v \in H_0^1(\Omega)$ be defined by the equation

$$(\nabla v, \nabla \varphi) = -(\nabla \tilde{B}_h q_h, \nabla \varphi) \quad \forall \varphi \in H_0^1(\Omega).$$

Then, v_h can be viewed as the Ritz projection of v. The estimate (3.11) implies that $\|\nabla v\|_{L^p(\Omega)} \le c \|\nabla \tilde{B}_h q_h\|_{L^p(\Omega)}$. Then, by Lemma 3.1, we obtain the estimate

$$\|\nabla v_h\|_{L^p(\Omega)} \le c \|\nabla \tilde{B}_h q_h\|_{L^p(\Omega)}, \quad 1 < p < \infty.$$

This implies that

$$\|\nabla B_h q_h\|_{L^p(\Omega)} \le \|\nabla v_h\|_{L^p(\Omega)} + \|\nabla \tilde{B}_h q_h\|_{L^p(\Omega)} \le c \|\nabla \tilde{B}_h q_h\|_{L^p(\Omega)}. \tag{3.12}$$

Therefore it remains to estimate the norm $\|\nabla \tilde{B}_h q_h\|_{L^p(\Omega)}$, which is localized to a strip S_h along the boundary of width h. For this purpose let $T \in \mathbb{T}_h$ be a cell of the decomposition of $\bar{\Omega}$, which nontrivially intersects the boundary: $\Gamma_T := T \cap \bar{\Omega} \neq \emptyset$. Then, by a standard argument employing transformations to a reference unit cell, there holds

$$\|\nabla \tilde{B}_h q_h\|_{L^p(T)}^p \le c h^{1-p} \|q_h\|_{L^p(\Gamma_T)}^p,$$

and summing this over all such cells belonging to the strip S_h,

$$\|\nabla \tilde{B}_h q_h\|_{L^p(S_h)} \le c h^{1/p-1} |q_h|_{L^p(\Gamma)}.$$

This proves the asserted estimate. □

LEMMA 3.4. *For the harmonic extensions $B \colon H^1(\Gamma) \to H^{3/2}(\Omega)$ and $B_h : V_h^\partial \to V_h$ there hold the error estimates*

$$\|\nabla (B - B_h) q_h\| \le c h^{1/2 - 1/p} |q_h|_{H^{1-1/p}(\Gamma)}, \tag{3.13}$$

$$(\psi, (B - B_h) q_h) \le c h^{2 - 1/p - 1/r} |q_h|_{H^{1-1/p}(\Gamma)} \|\psi\|_{L^r(\Omega)}, \tag{3.14}$$

for $q_h \in V_h^\partial$, $\psi \in L^r(\Omega)$, $2 \le r \le p < p_*^\Omega$. Particularly, there holds

$$\|(B - B_h) q_h\| \le c h^{3/2 - 1/p} |q_h|_{H^{1-1/p}(\Gamma)}. \tag{3.15}$$

Proof. Notice that $B_h q_h \in V_h$ is just the Ritz projection of $B q_h$ corresponding to the same boundary values $q_h \in V_h^\partial$,

$$(\nabla (B - B_h) q_h, \nabla \varphi_h) = 0 \quad \forall \varphi_h \in V_{h,0}, \quad (B - B_h) q_h|_\Gamma = 0.$$

(i) For the modified interpolation operator $\tilde{I}_h$ defined in (3.3) there holds $(\tilde{I}_h B - B_h) q_{h|\Gamma} = 0$. Hence,

$$\begin{aligned} \|\nabla (B - B_h) q_h\|^2 &= (\nabla (B - B_h) q_h, \nabla (B - \tilde{I}_h B) q_h) \\ &\le \|\nabla (B - B_h) q_h\| \|\nabla (B - \tilde{I}_h B) q_h\|. \end{aligned}$$

The interpolation estimate (3.3) together with the stability estimate (2.10) yields

$$\begin{aligned}\|\nabla(B-\tilde I_h B)q_h\| &\le ch^{1/2-1/p}\|Bq_h\|_{H^{3/2-1/p}(\Omega)}\\ &\le ch^{1/2-1/p}|q_h|_{H^{1-1/p}(\Gamma)},\end{aligned}$$

which proves the estimate (3.13).

(ii) For an arbitrary but fixed $\psi\in L^r(\Omega)$ let $w\in H_0^1(\Omega)$ be the solution of the auxiliary problem

$$-\Delta w=\psi \quad \text{in } \Omega, \quad w|_\Gamma=0$$

satisfying $w\in W^{2,r}(\Omega)$, for $2\le r<p_*^\Omega$, and $\|w\|_{W^{2,r}(\Omega)}\le c\|\psi\|_{L^r(\Omega)}$ (see Grisvard [9]). Then, with the Neumann projection $R_h^N: H^1(\Omega)\to V_h$ and observing $(B-B_h)q_h|_\Gamma=0$,

$$\begin{aligned}(\psi,(B-B_h)q_h) &= -(\Delta w,(B-B_h)q_h)=(\nabla w,\nabla(B-B_h)q_h)\\ &=(\nabla(w-R_h^N w),\nabla(B-B_h)q_h)+(\nabla R_h^N w,\nabla(B-B_h)q_h)\\ &=: \Lambda_1+\Lambda_2.\end{aligned}$$

The two terms Λ_1 and Λ_2 are estimated separately using the properties of B and R_h^N:

$$\begin{aligned}\Lambda_1 &= (\nabla(w-R_h^N w),\nabla(B-B_h)q_h)\\ &=(\nabla(w-R_h^N w),\nabla Bq_h)+(w-R_h^N w,B_hq_h)\\ &=\langle w-R_h^N w,\partial_n Bq_h\rangle-(w-R_h^N w,\Delta Bq_h)+(w-R_h^N w,B_hq_h)\\ &=-\langle R_h^N w,\partial_n Bq_h\rangle+(w-R_h^N w,B_hq_h).\end{aligned}$$

The first term on the right is estimated using the inverse estimate (3.1), the a priori estimate (2.14), and the error estimate (3.7) as follows:

$$\begin{aligned}-\langle R_h^N w,\partial_n Bq_h\rangle &\le |R_h^N w|_{H^{1/p}(\Gamma)}|\partial_n Bq_h|_{H^{-1/p}(\Gamma)}\\ &\le ch^{-1/p}|R_h^N w|\,|q_h|_{H^{1-1/p}(\Gamma)}\\ &\le ch^{-1/p}|w-R_h^N w|_{L^r(\Gamma)}|q_h|_{H^{1-1/p}(\Gamma)}\\ &\le ch^{2-1/r-1/p}\|w\|_{W^{2,r}(\Omega)}|q_h|_{H^{1-1/p}(\Gamma)}\\ &\le ch^{2-1/r-1/p}\|\psi\|_{L^r(\Omega)}|q_h|_{H^{1-1/p}(\Gamma)}.\end{aligned}$$

For the second term, by similar but simpler arguments we have

$$\begin{aligned}(w-R_h^N w,B_hq_h) &\le \|w-R_h^N w\|\|B_hq_h\|\\ &\le ch^2\|\psi\|_{L^r(\Omega)}|q_h|_{H^{1-1/p}(\Gamma)}.\end{aligned}$$

Thus,

$$\Lambda_1\le ch^{2-1/r-1/p}\|\psi\|_{L^r(\Omega)}|q_h|_{H^{1-1/p}(\Gamma)}.$$

Next, we denote by a_i the nodal points of the mesh $\mathbb{T}_h$ and by φ_h^i the corresponding nodal basis functions satisfying $\|\nabla\varphi_h^i\|_{L^2(\operatorname{supp}(\varphi_h^i))}\le c$. Then, using the properties of the harmonic extensions, we estimate as follows:

$$\begin{aligned}\Lambda_2 &= (\nabla R_h^N w, \nabla(B-B_h)q_h) = \sum_{a_i\in\bar\Omega} R_h^N w(a_i)(\nabla\varphi_h^i, \nabla(B-B_h)q_h)\\ &= \sum_{a_i\in\Gamma} R_h^N w(a_i)(\nabla\varphi_h^i, \nabla(B-B_h)q_h)\\ &\le \sum_{a_i\in\Gamma} |R_h^N w(a_i)|\|\nabla\varphi_h^i\|_{L^2(\mathrm{supp}(\varphi_h^i))}\|\nabla(B-B_h)q_h\|_{L^2(\mathrm{supp}(\varphi_h^i))}\\ &\le ch^{-1/2}\Big(\sum_{a_i\in\Gamma} h|R_h^N w(a_i)|^2\Big)^{1/2}\Big(\sum_{a_i\in\Gamma}\|\nabla(B-B_h)q_h\|^2_{L^2(\mathrm{supp}(\varphi_h^i))}\Big)^{1/2}\\ &\le ch^{-1/2}|R_h^N w - w|\|\nabla(B-B_h)q_h\|.\end{aligned}$$

Then, by the estimates (3.7) of Lemma 3.2 and the already proved estimate (3.13),

$$\begin{aligned}\Lambda_2 &\le ch^{-1/2}h^{2-1/r}\|w\|_{W^{2,r}(\Omega)}h^{1/2-1/p}|q_h|_{H^{1-1/p}(\Gamma)}\\ &\le ch^{2-1/r-1/p}\|\psi\|_{L^r(\Omega)}|q_h|_{H^{1-1/p}(\Gamma)}.\end{aligned}$$

Combining the estimates for Λ_1 and Λ_2, we obtain

$$(\psi, (B-B_h)q_h) \le ch^{2-1/p-1/r}\|\psi\|_{L^r(\Omega)}|q_h|_{H^{1-1/p}(\Gamma)},$$

which proves the estimate (3.14).

(iii) The L^2-norm estimate (3.15) follows from (3.14) by setting $\psi := (B-B_h)q_h$ and $r=2$. This completes the proof. □

4. Basic error estimates. In the following, we will use the quantity

$$\Sigma_p := |\hat q|_{H^{1-1/p}(\Gamma)} + \|f\| + \|\hat z\|_{W^{2,p}(\Omega)},$$

which is bounded for $2 \le p < p_*$, according to Lemma 2.10, with $p_* := \min\{p_*^d, p_*^\Omega\}$. The constants c_α appearing below may blow up for $p \to p_*$. We begin with the estimate of the reduced state error.

LEMMA 4.1. *For the reduced state error* $e_v = \hat v - \hat v_h$ *there holds*

$$\|e_v\| \le ch^2\|f\|. \tag{4.1}$$

Proof. The function $\hat v = \hat u - B\hat q \in H_0^1(\Omega)$ is the solution of the boundary value problem

$$-\Delta\hat v = f \quad \text{in } \Omega, \quad \hat v|_\Gamma = 0,$$

and $\hat v_h \in V_{h,0}$ denotes its Ritz projection satisfying

$$(\nabla e_v, \nabla\varphi_h) = 0, \quad \varphi_h \in V_{h,0}.$$

Since $f \in L^2(\Omega)$, we have $\hat v \in H^2(\Omega)$ and the estimate (3.6) of Lemma 3.1 yields the asserted error estimate. □

As starting point for the estimate of the control, we recall the perturbed Galerkin orthogonality equation (2.52),

$$\alpha\langle e_q, \chi_h\rangle + (\hat u - u_d, B\chi_h) - (\hat u_h - u_d, B_h\chi_h) = 0, \quad \chi_h \in V_h^\partial, \tag{4.2}$$

where $\hat{u} = \hat{v} + B\hat{q}$ and $\hat{u}_h = \hat{v}_h + B_h\hat{q}_h$. This can be rearranged into the form

$$\alpha\langle e_q, \chi_h\rangle = -(B\hat{q} - B_h\hat{q}_h, B_h\chi_h) - (\hat{v} - \hat{v}_h, B_h\chi_h) - (\hat{u} - u_d, (B - B_h)\chi_h). \tag{4.3}$$

THEOREM 4.2. *For the control error* $e_q := \hat{q} - \hat{q}_h$ *and the state error* $e_u := \hat{u} - \hat{u}_h$ *there holds the estimate*

$$|e_q| + \|e_u\| \le c_\alpha h^{1-1/p}\Sigma_p, \tag{4.4}$$

for $2 \le p < p_*$ *, where* $c_\alpha \approx 1 + \alpha^{-1}$ *.*

Proof. (i) We begin with the relation

$$\alpha|e_q|^2 = \alpha\langle e_q, \hat{q} - P_h^\partial\hat{q}\rangle + \alpha\langle e_q, P_h^\partial e_q\rangle.$$

Setting $\chi_h := P_h^\partial e_q = P_h^\partial\hat{q} - \hat{q}_h$ in (4.3) and rearranging terms, we obtain

$$\begin{aligned}
\alpha\langle e_q, P_h^\partial e_q\rangle &= -(B\hat{q} - B_h\hat{q}_h, B_hP_h^\partial e_q) - (\hat{v} - \hat{v}_h, B_hP_h^\partial e_q)\\
&\quad - (\hat{u} - u_d, (B - B_h)P_h^\partial e_q)\\
&= (B\hat{q} - B_h\hat{q}_h, (B - B_h)P_h^\partial\hat{q}) - (B\hat{q} - B_h\hat{q}_h, B(P_h^\partial\hat{q} - \hat{q}))\\
&\quad - (B\hat{q} - B_h\hat{q}_h, B\hat{q} - B_h\hat{q}_h) - (\hat{v} - \hat{v}_h, B_hP_h^\partial e_q)\\
&\quad - (\hat{u} - u_d, (B - B_h)P_h^\partial e_q).
\end{aligned}$$

Combining this with the first equation results in

$$\begin{aligned}
\alpha|e_q|^2 + \|B\hat{q} - B_h\hat{q}_h\|^2 &= \alpha\langle e_q, \hat{q} - P_h^\partial\hat{q}\rangle + (B\hat{q} - B_h\hat{q}_h, (B - B_h)P_h^\partial\hat{q})\\
&\quad - (B\hat{q} - B_h\hat{q}_h, B(P_h^\partial\hat{q} - \hat{q})) - (\hat{v} - \hat{v}_h, B_hP_h^\partial e_q)\\
&\quad - (\hat{u} - u_d, (B - B_h)P_h^\partial e_q).
\end{aligned} \tag{4.5}$$

The five terms on the right hand side of (4.5) will be treated separately.

First term: By the error estimate (3.4) for P_h^∂ ,

$$\begin{aligned}
\alpha\langle e_q, \hat{q} - P_h^\partial\hat{q}\rangle &\le \alpha|e_q|\,|\hat{q} - P_h^\partial\hat{q}| \le c\alpha h^{1-1/p}|e_q|\,|\hat{q}|_{H^{1-1/p}(\Gamma)}\\
&\le \tfrac{\alpha}{4}|e_q|^2 + c\alpha h^{2-2/p}|\hat{q}|^2_{H^{1-1/p}(\Gamma)}.
\end{aligned}$$

Second term: By the L^2 error estimate (3.15) and the stability estimate (3.4),

$$\begin{aligned}
(B\hat{q} - B_h\hat{q}_h, (B - B_h)P_h^\partial\hat{q}) &\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + \|(B - B_h)P_h^\partial\hat{q}\|^2\\
&\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + ch^{3-2/p}|P_h^\partial\hat{q}|^2_{H^{1-1/p}(\Gamma)}\\
&\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + ch^{3-2/p}|\hat{q}|^2_{H^{1-1/p}(\Gamma)}.
\end{aligned}$$

Third term: By the stability estimate (2.10) and the error estimate (3.4),

$$\begin{aligned}
-(B\hat{q} - B_h\hat{q}_h, B(P_h^\partial\hat{q} - \hat{q})) &\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + \|B(P_h^\partial\hat{q} - \hat{q})\|^2\\
&\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + c|P_h^\partial\hat{q} - \hat{q}|^2\\
&\le \tfrac14\|B\hat{q} - B_h\hat{q}_h\|^2 + ch^{2-2/p}|\hat{q}|^2_{H^{1-1/p}(\Gamma)}.
\end{aligned}$$

Fourth term: By the estimate (3.9) of Lemma 3.3, the L^2 stability of the projection P_h^∂ , and the estimate (4.1) of Lemma 4.1,

$$\begin{aligned}
-(\hat{v} - \hat{v}_h, B_hP_h^\partial e_q) &\le \|\hat{v} - \hat{v}_h\|\|B_hP_h^\partial e_q\| \le c\|\hat{v} - \hat{v}_h\|\,|P_h^\partial e_q|\\
&\le c\|\hat{v} - \hat{v}_h\|\,|e_q| \le \tfrac{c}{\alpha}\|\hat{v} - \hat{v}_h\|^2 + \tfrac{\alpha}{4}|e_q|^2\\
&\le \tfrac{c}{\alpha}h^4\|f\|^2 + \tfrac{\alpha}{4}|e_q|^2.
\end{aligned}$$

Fifth term: We recall that $\hat{z} \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies $-\Delta \hat{z} = \hat{u} - u_d$ in Ω. Hence, observing that $(B - B_h)P_h^\partial e_q|_\Gamma = 0$, with $p' = p(p-1)^{-1} \leq 2$, and using the properties of the harmonic extension operators B and B_h, we deduce that

$$\begin{aligned}
-(\hat{u} - u_d, (B - B_h)P_h^\partial e_q) &= (\Delta \hat{z}, (B - B_h)P_h^\partial e_q) \\
&= -(\nabla \hat{z}, \nabla (B - B_h)P_h^\partial e_q) + \langle \partial_n z, (B - B_h)P_h^\partial e_q \rangle \\
&= (\nabla(\hat{z} - I_h \hat{z}), \nabla B_h P_h^\partial e_q).
\end{aligned}$$

Further, using the interpolation error estimate (3.2), the a priori estimate (3.10) of Lemma 3.3, and the L^2 stability of the projection P_h^∂

$$\begin{aligned}
-(\hat{u} - u_d, (B - B_h)P_h^\partial e_q) &\leq \|\nabla(\hat{z} - I_h \hat{z})\|_{L^p(\Omega)} \|\nabla B_h P_h^\partial e_q\|_{L^{p'}(\Omega)} \\
&\leq ch \|\hat{z}\|_{W^{2,p}(\Omega)} h^{1/p' - 1} |P_h^\partial e_q| \\
&= ch^{1-1/p} \|\hat{z}\|_{W^{2,p}(\Omega)} |P_h^\partial e_q| \\
&\leq c\alpha^{-1} h^{2-2/p} \|\hat{z}\|^2_{W^{2,p}(\Omega)} + \tfrac{\alpha}{4} |e_q|^2.
\end{aligned}$$

Collecting the foregoing estimates, we obtain

$$\begin{aligned}
\alpha |e_q|^2 + \|B\hat{q} - B_h \hat{q}_h\|^2 \leq \tfrac{3}{4}\alpha |e_q|^2 &+ c(1+\alpha) h^{2-2/p} |\hat{q}|^2_{H^{1-1/p}(\Gamma)} + \tfrac{1}{2}\|B\hat{q} - B_h \hat{q}_h\|^2 \\
&+ c\alpha^{-1} h^4 \|f\|^2 + c\alpha^{-1} h^{2-2/p} \|\hat{z}\|^2_{W^{2,p}(\Omega)},
\end{aligned}$$

and absorbing terms into the left hand side,

$$\alpha |e_q|^2 \leq ch^{2-2/p} |\hat{q}|^2_{H^{1-1/p}(\Gamma)} + c\alpha^{-1} \big\{ h^4 \|f\|^2 + h^{2-2/p} \|\hat{z}\|^2_{W^{2,p}(\Omega)} \big\}.$$

This proves the asserted estimate of $|e_q|$.

(ii) Further, observing that $e_u = e_v + B\hat{q} - B_h \hat{q}_h$ and the estimate (4.1) of Lemma 4.1 for $\|e_v\|$, we obtain the estimate of $\|e_u\|$. ∎

REMARK 5. The estimate of the control error provided by Theorem 4.2 is order-optimal with respect to the regularity of the solution, but that for the state error is only suboptimal. The latter will be improved in the next section.

COROLLARY 4.3. *The discrete controls admit the uniform bound*

$$|\hat{q}_h|_{H^{1-1/p}(\Gamma)} \leq c_\alpha \Sigma_p, \quad 2 \leq p < p_*. \tag{4.6}$$

Proof. Using the foregoing results together with the estimate (3.4) for the L^2 projection P_h^∂, we estimate as follows:

$$\begin{aligned}
|\hat{q}_h|_{H^{1-1/p}(\Gamma)} &\leq |\hat{q}_h - P_h^\partial \hat{q}|_{H^{1-1/p}(\Gamma)} + |P_h^\partial \hat{q} - \hat{q}|_{H^{1-1/p}(\Gamma)} + |\hat{q}|_{H^{1-1/p}(\Gamma)} \\
&\leq ch^{-1+1/p} |\hat{q}_h - P_h^\partial \hat{q}| + c|\hat{q}|_{H^{1-1/p}(\Gamma)} \\
&\leq ch^{-1+1/p} \big\{ |e_q| + |\hat{q} - P_h^\partial \hat{q}| \big\} + c|\hat{q}|_{H^{1-1/p}(\Gamma)} \\
&\leq ch^{-1+1/p} |e_q| + c|\hat{q}|_{H^{1-1/p}(\Gamma)} \\
&\leq c_\alpha \big\{ |\hat{q}|_{H^{1-1/p}(\Gamma)} + \|f\| + \|\hat{z}\|_{W^{2,p}(\Omega)} \big\}.
\end{aligned}$$

This implies the asserted estimate. ∎

5. Improved error estimates. The orders of convergence derived in the preceding section for the state $\hat{u}$ may not be optimal as is demonstrated by the numerical results presented in Section 6, below. The key to improved error estimates for the state is the proof of higher order error estimates for the control in norms weaker than the L^2 norm. To this end, we will employ duality arguments based on the KKT system. The modified KKT system (2.38), (2.39), (2.40) can be written in compact form for the triplet $\hat{X} := \{\hat{v}, \hat{q}, \hat{z}\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$ as follows:

$$A(\hat{X}, \Phi) = F(\Phi), \tag{5.1}$$

for all $\Phi = \{\varphi^z, \varphi^q, \varphi^v\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$, with the bilinear form

$$A(\hat{X}, \Phi) := (\nabla \hat{v}, \nabla \varphi^v) + \alpha \langle \hat{q}, \varphi^q \rangle + (\hat{v} + B\hat{q}, B\varphi^q) + (\nabla \hat{z}, \nabla \varphi^z) - (\hat{v} + B\hat{q}, \varphi^z)$$

and the right hand side

$$F(\Phi) := (f, \varphi^v) - (u_d, \varphi^z) + (u_d, B\varphi^q).$$

For a given linear functional $J(\cdot)$ let $W = \{w^v, w^q, w^z\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$ be the solution of the dual problem

$$A(\Psi, W) = J(\Psi) \quad \forall \Psi = \{\psi^z, \psi^q, \psi^v\}. \tag{5.2}$$

For $J(\Psi) = J_u(\psi^v) + J_q(\psi^q) + J_z(\psi^z)$ this is equivalent to the system

$$(\nabla \psi^v, \nabla w^v) = J_u(\psi^v) \quad \forall \psi^v \in H_0^1(\Omega), \tag{5.3}$$

$$\alpha \langle \psi^q, w^q \rangle + (B\psi^q, Bw^q) - (B\psi^q, w^v) = J_q(\psi^q) \quad \forall \psi^q \in H^{1/2}(\Gamma), \tag{5.4}$$

$$(\nabla \psi^z, \nabla w^z) - (\psi^z, w^v) + (\psi^z, Bw^q) = J_z(\psi^z) \quad \forall \psi^z \in H_0^1(\Omega). \tag{5.5}$$

In the special case $J(\Psi) = J_q(\psi^q)$, the first equation (5.3) has the unique solution $w^v = 0$, and we obtain the triangular system

$$\alpha \langle \psi^q, w^q \rangle + (B\psi^q, Bw^q) = J_q(\psi^q) \quad \forall \psi^q \in H^{1/2}(\Gamma), \tag{5.6}$$

$$(\nabla \psi^z, \nabla w^z) + (\psi^z, Bw^q) = 0 \qquad \forall \psi^z \in H_0^1(\Omega). \tag{5.7}$$

By coercivity arguments it is easily seen that for $J_q(\cdot) \in L^2(\Gamma)^*$ there is a unique solution $\{w^q, w^z\} \in L^2(\Gamma) \times H_0^1(\Omega)$.

LEMMA 5.1. *For $J_q(\Psi) = (B\psi^q, \psi)$ with a fixed $\psi \in L^r(\Omega)$, $2 \le r < p_*^\Omega$, the solution $\{w^q, w^z\}$ of the dual system (5.6), (5.7) has the regularity $\{w^q, w^z\} \in H^{1-1/r}(\Gamma) \times W^{2,r}(\Omega)$ and there holds the a priori estimate*

$$|w^q|_{H^{1-1/r}(\Gamma)} + \|w^z\|_{W^{2,r}(\Omega)} \le c\alpha^{-1} \|\psi\|_{L^r(\Omega)}. \tag{5.8}$$

Proof. For $Bw^q \in L^2(\Omega)$, we infer that $w^z \in H_0^1(\Omega) \cap H^2(\Omega)$. Since (5.6) also holds for $\psi^q \in L^2(\Gamma)$, we can test with $\psi^q = w^q$, which gives us

$$\alpha |w^q| + \|Bw^q\| \le c\|\psi\|.$$

Further, by elliptic regularity, we have $\|w^z\|_{H^2(\Omega)} \le c\|\psi\|$. Next, we employ a duality argument. Let $\eta \in H_0^1(\Omega)$ be the weak solution of the auxiliary problem

$$-\Delta \eta = \psi - Bw^q \quad \text{in } \Omega.$$

Since $Bw^q - \psi \in L^2(\Omega)$, we have $\eta \in H^2(\Omega)$, and $\partial_n \eta|_\Gamma \in H^s(\Gamma)$ for $0 \le s < 1/2$ by Lemma 2.3. Further,

$$\|\eta\|_{H^2(\Omega)} + |\partial_n \eta|_{H^s(\Gamma)} \le c\|\psi\|.$$

Then, from

$$\alpha\langle \chi, w^q\rangle = (B\chi, \psi - Bw^q) = -(B\chi, \Delta\eta) = (\nabla B\chi, \nabla\eta) + \langle \chi, \partial_n\eta\rangle = \langle \chi, \partial_n \eta\rangle$$

we infer that also $w^q = \alpha^{-1}\partial_n\eta \in H^s(\Gamma)$. In view of the Lemma 2.2 this implies that $Bw^q \in H^{s+1/2}(\Omega)$. As s can be taken arbitrarily close to $1/2$, by the Sobolev embedding theorem we conclude that $Bw^q \in L^r(\Omega)$ for $r \ge 2$ as considered and $\|Bw^q\|_{L^r(\Omega)} \le c\|\psi\|$. Now, this implies that $\eta \in H_0^1(\Omega) \cap W^{2,r}(\Omega)$ and consequently, in virtue of Lemma 2.3, $w^q = \alpha^{-1}\partial_n\eta \in W^{1-1/r,r}(\Gamma) \subset H^{1-1/r}(\Gamma)$ and $\|w^q\|_{H^{1-1/r}(\Gamma)} \le c\alpha^{-1}\|\psi\|_{L^r(\Omega)}$. Finally, again by elliptic regularity theory, we obtain $w^z \in H_0^1(\Omega) \cap W^{2,r}(\Omega)$ and $\|w^z\|_{W^{2,r}(\Omega)} \le c\alpha^{-1}\|\psi\|_{L^r(\Omega)}$. This completes the proof. ∎

THEOREM 5.2. *For the control error* $e_q := \hat{q} - \hat{q}_h$ *and any* $\psi \in L^r(\Omega)$ *there holds*

$$(Be_q, \psi) \le c_\alpha^2 h^{2-1/p-1/r} \Sigma_p \|\psi\|_{L^r(\Omega)}, \tag{5.9}$$

for $2 \le p < p_*$ *(depending on the regularity of the data and the domain) and* $2 \le r < p_*^\Omega$ *(depending on the regularity of the domain), where* $c_\alpha^2 \approx 1 + \alpha^{-2}$.

Proof. We use the dual problem described above with $\psi \in L^p(\Omega)$. Taking the test pair $\{\psi^q, \psi^z\} = \{e_q, e_v\}$ in the equations (5.6) and (5.7) gives us

$$\alpha\langle e_q, w^q\rangle + (Be_q, Bw^q) = (Be_q, \psi), \tag{5.10}$$

$$(\nabla e_v, \nabla w^z) + (e_v, Bw^q) = 0. \tag{5.11}$$

Then, subtracting the L^2 projection $P_h^\partial w^q \in V_h^\partial$ in (5.10), yields

$$\begin{aligned}(Be_q, \psi) = \alpha\langle e_q, w^q - P_h^\partial w^q\rangle + (Be_q, B(w^q - P_h^\partial w^q))\\ + \alpha\langle e_q, P_h^\partial w^q\rangle + (Be_q, BP_h^\partial w^q).\end{aligned} \tag{5.12}$$

We recall the Galerkin orthogonality relations (2.51), (2.52), (2.53):

$$(\nabla e_v, \nabla \varphi_h) = 0 \quad \forall \varphi_h \in V_{h,0}, \tag{5.13}$$

$$\alpha\langle e_q, \chi_h\rangle + (\hat{v} + B\hat{q} - u_d, B\chi_h) - (\hat{v}_h + B_h\hat{q}_h - u_d, B_h\chi_h) = 0 \quad \forall \chi_h \in V_h^\partial, \tag{5.14}$$

$$(\nabla e_z, \nabla \psi_h) - (e_v + B\hat{q} - B_h\hat{q}_h, \psi_h) = 0 \quad \forall \psi_h \in V_{h,0}. \tag{5.15}$$

We also set $\chi_h = P_h^\partial w^q$ in (5.14) and rearrange terms to obtain

$$\begin{aligned}
0 &= \alpha\langle e_q, P_h^\partial w^q\rangle + (\hat v + B\hat q - u_d, BP_h^\partial w^q) - (\hat v_h + B_h\hat q_h - u_d, B_hP_h^\partial w^q)\\
&= \alpha\langle e_q, P_h^\partial w^q\rangle + (Be_q, BP_h^\partial w^q) - (Be_q, (B-B_h)P_h^\partial w^q) - (Be_q, B_hP_h^\partial w^q)\\
&\quad + (\hat v + B\hat q - u_d, BP_h^\partial w^q) - (\hat v_h + B_h\hat q_h - u_d, B_hP_h^\partial w^q)\\
&= \alpha\langle e_q, P_h^\partial w^q\rangle + (Be_q, BP_h^\partial w^q) - (Be_q, (B-B_h)P_h^\partial w^q) - (Be_q, B_hP_h^\partial w^q)\\
&\quad + (\hat v + B\hat q - u_d, (B-B_h)P_h^\partial w^q) + (\hat v + B\hat q - u_d, B_hP_h^\partial w^q)\\
&\quad - (\hat v_h + B_h\hat q_h - u_d, B_hP_h^\partial w^q)\\
&= \alpha\langle e_q, P_h^\partial w^q\rangle + (Be_q, BP_h^\partial w^q) - (Be_q, (B-B_h)P_h^\partial w^q)\\
&\quad + (e_v + B\hat q - B_h\hat q_h - Be_q, B_hP_h^\partial w^q) + (\hat u - u_d, (B-B_h)P_h^\partial w^q)\\
&= \alpha\langle e_q, P_h^\partial w^q\rangle + (Be_q, BP_h^\partial w^q) - (Be_q, (B-B_h)P_h^\partial w^q)\\
&\quad + (e_v, B_hP_h^\partial w^q) + ((B-B_h)\hat q_h, B_hP_h^\partial w^q) + (\hat u - u_d, (B-B_h)P_h^\partial w^q).
\end{aligned}$$

Combining this with (5.12) gives us

$$\begin{aligned}
(Be_q, \psi) &= \alpha\langle e_q, w^q - P_h^\partial w^q\rangle + (Be_q, B(w^q - P_h^\partial w^q))\\
&\quad + (Be_q, (B-B_h)P_h^\partial w^q) - (e_v, B_hP_h^\partial w^q)\\
&\quad - ((B-B_h)\hat q_h, B_hP_h^\partial w^q) - (\hat u - u_d, (B-B_h)P_h^\partial w^q).
\end{aligned}\tag{5.16}$$

The six terms on the right hand side of (5.16) will be treated separately with the sixth term being the most difficult one.

First term: By the properties of the L^2 projection P_h^∂ and the error estimate (3.4), for $2 \le p \le p_*$, $2 \le r \le p_*^\Omega$,

$$\begin{aligned}
\alpha\langle e_q, w^q - P_h^\partial w^q\rangle &= \alpha\langle \hat q - P_h^\partial \hat q, w^q - P_h^\partial w^q\rangle\\
&\le \alpha|\hat q - P_h^\partial \hat q|\,|w^q - P_h^\partial w^q|\\
&\le c\alpha h^{1-1/p}|\hat q|_{H^{1-1/p}(\Gamma)} h^{1-1/r}|w^q|_{H^{1-1/r}(\Gamma)}\\
&\le c\alpha h^{2-1/p-1/r}\Sigma_p|w^q|_{H^{1-1/r}(\Gamma)}.
\end{aligned}$$

Second term: By the stability estimate (2.10), the error estimate (3.4), and the estimate (4.4) of Theorem 4.2, for $2 \le p \le p_*^d$, $2 \le r \le p_*^\Omega$,

$$\begin{aligned}
(Be_q, B(w^q - P_h^\partial w^q)) &\le \|Be_q\|\|B(w^q - P_h^\partial w^q)\|\\
&\le c|e_q|\,|w^q - P_h^\partial w^q|\\
&\le c_\alpha h^{1-1/p}\Sigma_p h^{1-1/r}|w^q|_{H^{1-1/r}(\Gamma)}\\
&= c_\alpha h^{2-1/p-1/r}\Sigma_p|w^q|_{H^{1-1/r}(\Gamma)}.
\end{aligned}$$

Third term: By the stability estimate (2.10), the error estimate (3.15) of Lemma 3.4, the result (4.4) of Theorem 4.2, and the stability estimate (3.4), for $2 \le p \le p_*^d$,

$$\begin{aligned}
(Be_q, (B-B_h)P_h^\partial w^q) &\le \|Be_q\|\|(B-B_h)P_h^\partial w^q\|\\
&\le c|e_q|h^{3/2-1/r}|P_h^\partial w^q|_{H^{1-1/r}(\Gamma)}\\
&\le c_\alpha h^{1-1/p}\Sigma_p h^{3/2-1/r}|P_h^\partial w^q|_{H^{1-1/r}(\Gamma)}\\
&\le c_\alpha h^{5/2-1/p-1/r}\Sigma_p|w^q|_{H^{1-1/r}(\Gamma)}.
\end{aligned}$$

Fourth term: By the error estimate (4.1) of Lemma 4.1, the stability estimates (3.9) of Lemma 3.3 and (3.4),

$$\begin{aligned}-(e_v, B_h P_h^\partial w^q) &\le \|e_v\| \|B_h P_h^\partial w^q\| \\ &\le ch^2 \|f\| \, |P_h^\partial w^q| \\ &\le ch^2 \Sigma_p |w^q|_{H^{1-1/r}(\Gamma)}.\end{aligned}$$

Fifth term: By the error estimate (3.14) of Lemma 3.4, the stability estimates (3.8), (3.9) of Lemma 3.3, the stability estimate (4.6) of Corollary 4.3, the stability estimate (3.4), for $2 \le r \le p_*^\Omega$ and $2 \le p \le p_*$,

$$\begin{aligned}-((B - B_h)\hat{q}_h, B_h P_h^\partial w^q) &\le ch^{2-1/r-1/p} |\hat{q}_h|_{H^{1-1/p}(\Gamma)} \|B_h P_h^\partial w^q\|_{L^r(\Omega)} \\ &\le ch^{2-1/r-1/p} |\hat{q}_h|_{H^{1-1/p}(\Gamma)} \|B_h P_h^\partial w^q\|_{H^1(\Omega)} \\ &\le ch^{2-1/r-1/p} |\hat{q}_h|_{H^{1-1/p}(\Gamma)} |P_h^\partial w^q|_{H^{1/2}(\Gamma)} \\ &\le c_\alpha h^{2-1/r-1/p} \Sigma_p |w^q|_{H^{1-1/r}(\Gamma)}.\end{aligned}$$

Sixth term: To estimate the sixth term, we recall that $\hat{z} \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies $-\Delta \hat{z} = \hat{u} - u_d$ in Ω. Hence, we can use the error estimate (3.14) of Lemma 3.4 and the stability estimate (3.4) to obtain, for $2 \le r \le p_*^\Omega$, $2 \le p \le p_*$,

$$\begin{aligned}-(\hat{u} - u_d, (B - B_h) P_h^\partial w^q) &\le ch^{2-1/p-1/r} \|\hat{z}\|_{W^{2,p}(\Omega)} |P_h^\partial w^q|_{H^{1-1/r}(\Gamma)} \\ &\le ch^{2-1/p-1/r} \Sigma_p |w^q|_{H^{1-1/r}(\Gamma)}.\end{aligned}$$

Combining all these estimates gives us

$$(Be_q, \psi) \le c_\alpha h^{2-1/p-1/r} \Sigma_p |w^q|_{H^{1-1/r}(\Gamma)},$$

and hence in view of Lemma 5.1,

$$(Be_q, \psi) \le c_\alpha^2 h^{2-1/p-1/r} \Sigma_p \|\psi\|_{L^r(\Omega)}.$$

This completes the proof. ◻

COROLLARY 5.3. *For the control error* $e_q := \hat{q} - \hat{q}_h$ *and the state error* $e_u := \hat{u} - \hat{u}_h$ *there holds*

$$|e_q|_{\tilde{H}^{-1/2}(\Gamma)} + \|e_u\| \le c_\alpha^2 h^{3/2-1/p} \Sigma_p, \tag{5.17}$$

for $2 \le p < p_*$ *, where* $c_\alpha^2 \approx 1 + \alpha^{-2}$ *.*

Proof. (i) Observing that Be_q is harmonic, by the trace extimate (2.7), we have

$$|e_q|_{\tilde{H}^{-1/2}(\Gamma)} \le c\|Be_q\|.$$

Taking $\psi := Be_q$ in the estimate (5.9) of Theorem 5.2 for $r = 2$, we obtain

$$\|Be_q\| \le c_\alpha^2 h^{3/2-1/p} \Sigma_p, \tag{5.18}$$

which implies the first part of the assertion.
(ii) From the identity

$$(e_u, \psi) = (e_v, \psi) + ((B - B_h)\hat{q}_h, \psi) + (Be_q, \psi), \tag{5.19}$$

we conclude that

$$\|e_u\| \le \|e_v\| + \|(B-B_h)\hat q_h\| + \|Be_q\|.$$

Hence, by the estimate (4.1) of Lemma 4.1, the estimate (3.15) of Lemma 3.4, and the just proven estimate (5.18),

$$\|e_u\| \le ch^2\|f\| + ch^{3/2-1/p}|\hat q_h|_{H^{1-1/p}(\Gamma)} + c_\alpha^2 h^{3/2-1/p}\Sigma_p.$$

In view of the estimate (4.6), this implies the second part of the assertion. □

COROLLARY 5.4. *For the primal state error* $e_u := \hat u - \hat u_h$ *and the adjoint state error* $e_z := \hat z - \hat z_h$ *there holds*

$$\|e_u\|_{H^{-1}(\Omega)} + \|e_z\| \le c_\alpha^2 h^{2-1/p-1/r}\Sigma_p, \tag{5.20}$$

for $2 \le p < p_*$ *and* $2 \le r < p_*^\Omega$, *where* $c_\alpha^2 \approx 1 + \alpha^{-2}$.

Proof. (i) We recall the identity (5.19). By Lemma 4.1, Lemma 3.4 and Theorem 5.2, we obtain

$$(e_u,\psi) \le c\big\{h^2\|f\| + h^{2-1/r-1/p}|\hat q_h|_{H^{1-1/p}(\Gamma)} + c_\alpha^2 h^{2-1/p-1/r}\Sigma_p\big\}\|\psi\|_{L^r(\Omega)},$$

for $2 \le p \le p_*$ and $2 \le r < p_*^\Omega$. By arguments already used before, this implies

$$\|e_u\|_{H^{-1}(\Omega)} = \sup_{\psi\in H^1(\Omega)} \frac{(e_u,\psi)}{\|\psi\|_{H^1(\Omega)}} \le c_\alpha^2 h^{2-1/p-1/r}\Sigma_p.$$

(ii) For proving the error estimate of the adjoint state, we recall the equation (2.53) in the form

$$(\nabla e_z, \nabla\psi_h) = (e_u,\psi_h), \quad \psi_h \in V_{h,0}. \tag{5.21}$$

The adjoint state $\hat z \in H_0^1(\Omega)$ is determined by the boundary value problem

$$-\Delta \hat z = \hat u - u_d \ \text{ in } \Omega, \quad \hat z|_\Gamma = 0,$$

and has the regularity $\hat z \in W^{2,p}(\Omega)$, with $2 \le p < p_*$. Let $w \in H_0^1(\Omega)\cap H^2(\Omega)$ be the solution of the auxiliary problem

$$-\Delta w = e_z \ \text{ in } \Omega, \quad w|_\Gamma = 0,$$

satisfying $\|w\|_{H^2(\Omega)} \le c\|e_z\|$. Then, using (2.53), we conclude

$$\begin{aligned}
\|e_z\|^2 &= (\nabla e_z, \nabla(w - R_h^D w)) + (\nabla e_z, \nabla R_h^D w)\\
&= (\nabla(\hat z - I_h\hat z), \nabla(w - R_h^D w)) + (e_u, R_h^D w)\\
&\le \|\nabla(\hat z - I_h \hat z)\|\|\nabla(w - R_h^D w)\| + \|e_u\|_{H^{-1}(\Omega)}\|R_h^D w\|_{H^1(\Omega)}\\
&\le ch^2\|\hat z\|_{H^2(\Omega)}\|w\|_{H^2(\Omega)} + c_\alpha^2 h^{2-1/p-1/r}\Sigma_p\|w\|_{H^2(\Omega)}\\
&\le c\big\{h^2 + c_\alpha^2 h^{2-1/p-1/r}\big\}\Sigma_p\|e_z\|.
\end{aligned}$$

This proves the asserted estimate. □

COROLLARY 5.5. *For the primal state error* $e_u := \hat u - \hat u_h$ *and the control error* $e_q := \hat q - \hat q_h$ *there holds*

$$|(e_u,1)| + |\langle e_q, 1\rangle| \le c_\alpha^2 h^{2-1/p-1/r}\Sigma_p, \tag{5.22}$$

for $2 \le p < p_*$ *and* $2 \le r < p_*^\Omega$, *where* $c_\alpha^2 \approx 1 + \alpha^{-2}$.

Proof. In view of

$$|(e_u, 1)| \le |\Omega|^{1/2} \sup_{\varphi \in H^1(\Omega)} \frac{(e_u, \varphi)}{\|\varphi\|_{H^1(\Omega)}} = |\Omega|^{1/2} \|e_u\|_{H^{-1}(\Omega)},$$

the first part of the asserted estimate follows from Corollary 5.4. Next, we recall the Galerkin orthogonality relation (2.52) in the form

$$\alpha \langle e_q, \chi_h \rangle = -(\hat{u} - u_d, B\chi_h) + (\hat{u}_h - u_d, B_h \chi_h) \quad \forall \chi_h \in V_h^\partial.$$

Hence observing that the function $\chi_h \equiv 1$ satisfies $\chi_h \in V_h^\partial$ and $B\chi_h \equiv B_h \chi_h \equiv 1$, it follows that

$$\alpha \langle e_q, 1 \rangle = -(e_u, 1).$$

This implies the second part of the asserted estimate. □

Remark 6. The numerical experiments shown in the next section indicate that for the adjoint state there may hold the improved error estimate

$$\|e_z\| \le c_\alpha^2 h^{2-1/r} \Sigma_p, \quad 2 \le r < p_*^\Omega. \tag{5.23}$$

6. Numerical tests. In this section, we present some numerical results obtained for the optimization problem (1.1), (1.2) by the discretization described above. The purpose is to clarify the convergence rates to be expected for several configurations. For the computation the software libraries GASCOIGNE [8] and RoDoBo [14] have been used. Three different configurations have been considered in order to illustrate the sharpness of our theoretically derived error estimates:

1. Regular domain (unit square) and known analytic solution.
2. General domain with $\omega_{\max} = \frac{5}{6}\pi$ and unknown solution.
3. Regular domain (unit square) and "singular" data.

6.1. Example on unit square with known analytic solution. The domain is the unit square $\Omega = (0,1)^2$ and the data is chosen as

$$f = -\tfrac{4}{\alpha}, \quad u_d = -\left(2 + \tfrac{1}{\alpha}\right)(x(1-x) + y(1-y)), \quad \alpha = 0.01,$$

such that the optimal solution is given by $\hat{q} = -\frac{1}{\alpha}(x(1-x) + y(1-y))$, $\hat{u} = -\frac{1}{\alpha}(x(1-x) + y(1-y))$ and $\hat{z} = xy(1-x)(1-y)$. The obtained results are presented in Tables 6.1, 6.2 and 6.3. The test on uniform cartesian meshes shows second-order convergence for all quantities which is probably due to superapproximation effects which are not captured by our analysis. These effects disappear on irregular meshes and the observed orders of convergence agree well with the theoretically predicted rates.

TABLE 6.1

Example with known analytic solution: convergence rates for a sequence of equidistant cartesian meshes with $h_n = 2^{-n}\sqrt{2}$

	$\lvert \hat{q} - \hat{q}_h \rvert$		$\lVert \hat{u} - \hat{u}_h \rVert$		$\lVert \hat{z} - \hat{z}_h \rVert$	
# cells	error	rate	error	rate	error	rate
64	4.81e-01	2.00	1.74e-02	2.00	3.04e-04	2.10
256	1.21e-01	2.00	4.25e-03	2.04	7.49e-05	2.02
1 024	3.02e-02	2.00	1.04e-03	2.03	1.87e-05	2.00
4 096	7.56e-03	2.00	2.58e-04	2.01	4.66e-06	2.00
16 384	1.89e-03	2.00	6.45e-05	2.00	1.17e-06	2.00
65 536	4.73e-04	2.00	1.61e-05	2.00	2.91e-07	2.00
expected		1.00		1.50		2.00

TABLE 6.2

Example with known analytic solution: convergence rates for a sequence of tensor-product meshes with 10% *random shift of interior nodal points after each uniform refinement step*

	$\lvert \hat{q} - \hat{q}_h \rvert$		$\lVert \hat{u} - \hat{u}_h \rVert$		$\lVert \hat{z} - \hat{z}_h \rVert$	
# cells	error	rate	error	rate	error	rate
64	4.91e-01	2.00	4.60e-02	2.00	3.71e-04	1.95
256	1.30e-01	1.92	1.30e-02	1.82	8.19e-05	2.18
1 024	4.53e-02	1.52	4.46e-03	1.54	2.00e-05	2.03
4 096	2.69e-02	0.75	1.98e-03	1.17	4.98e-06	2.01
16 384	1.74e-02	0.63	8.89e-04	1.16	1.24e-06	2.01
65 536	9.65e-03	0.85	3.48e-04	1.35	3.11e-07	1.99
expected		1.00		1.50		2.00

TABLE 6.3

Example with known analytic solution: convergence rates for the mean values on a sequence of tensor-product meshes with 10% *random shift of interior nodal points after each uniform refinement step*

	$\langle \hat{q} - \hat{q}_h, 1 \rangle$		$(\hat{u} - \hat{u}_h, 1)$	
# cells	error	rate	error	rate
64	-2.66e+00	2.08	2.66e-02	2.08
256	-6.54e-01	2.02	6.54e-03	2.02
1 024	-1.65e-01	1.99	1.65e-03	1.99
4 096	-4.10e-02	2.01	4.11e-04	2.01
16 384	-1.02e-02	2.00	1.02e-04	2.01
65 536	-2.51e-03	2.03	2.46e-05	2.05
expected		2.00		2.00

6.2. Example on general polygonal domain with unknown solution. Next, we test the convergence rates for a domain with maximum interior angle $\omega_{\max} = \frac{5}{6}\pi$ (see Figure 6.1) where the optimal solution has only reduced regularity. The data is taken as $\alpha = 1$ and

$$f = 1, \quad u_d = \begin{cases} -1 & \text{for } 0 \le y < 0.5 \\ 1 & \text{for } 0.5 \le y \le 1 \end{cases}.$$

Since $\hat{u} - u_d \in L^\infty(\Omega)$ the adjoint state satisfies $\hat{z} \in W^{2,p}(\Omega)$ for $2 \le p < p_*^\Omega$, where in this case $p_*^\Omega = 2\omega_{\max}(2\omega_{\max} - \pi)^{-1} = \frac{5}{2}$. The results obtained for this configuration are shown in Table 6.4. The "reference solution" has been calculated on a very fine mesh with more than 10^6 cells.

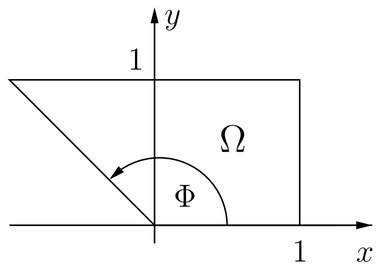

Fig. 6.1. *General domain with* $\omega_{\max} = \frac{5}{6}\pi$

Table 6.4
Results for a domain with maximum interior angle $\omega_{\max} = \frac{5}{6}\pi$ *: convergence rates on a sequence of irregular meshes*

$\omega_{\max} = \frac{5}{6}\pi$	$\vert\hat{q} - \hat{q}_h\vert$		$\Vert\hat{u} - \hat{u}_h\Vert$		$\Vert\hat{z} - \hat{z}_h\Vert$	
# cells	error	rate	error	rate	error	rate
64	4.40e-02	0.99	9.98e-03	1.50	2.34e-03	1.70
256	2.42e-02	0.87	4.04e-03	1.31	6.95e-04	1.75
1 024	1.47e-02	0.72	1.72e-03	1.23	2.43e-04	1.52
4 096	8.43e-03	0.80	7.25e-04	1.25	7.21e-05	1.75
16 384	5.35e-03	0.65	3.26e-04	1.15	2.21e-05	1.71
65 536	3.40e-03	0.65	1.47e-04	1.15	6.76e-06	1.71
expected		0.60		1.10		1.60

6.3. Example on unit square with irregular data. The following example has been adopted from Casas/Raymond [5]. The domain again is the unit square $\Omega = (0,1)^2$, $f = 0$ and $u_d = (x^2+y^2)^{-\frac{1}{3}}$. Hence $p_*^\Omega = \infty$. Since u_d has a singularity at the boundary Γ, such that $u_d \in L^p(\Omega)$ for $2 \le p < 3$ but $u_d \notin L^3(\Omega)$, the optimal solution has only reduced regularity $\{\hat{u}, \hat{q}, \hat{z}\} \in H^{\frac{3}{2}-\frac{1}{p}}(\Omega) \times W^{1-\frac{1}{p},p}(\Gamma) \times W^{2,p}(\Omega)$, with $2 \le p < p_*^d = 3$. Hence, acccording to our theory, we expect the errors for $\hat{q}$, $\hat{u}$ and $\hat{z}$ to converge with the rates $\approx 1 - \frac{1}{p_*^d} = \frac{2}{3}$, $\approx \frac{3}{2} - \frac{1}{p_*^d} = \frac{7}{6}$, and $\approx 2 - \frac{1}{p_*^d} - \frac{1}{p_*^\Omega} = \frac{5}{3}$, respectively. The results are shown in Table 6.5. The orders of convergence observed for e_q and e_u are in reasonable agreement with the theoretically predicted ones, while that for e_z seems to be too high.

Table 6.5
Results for irregular data: convergence rates on a sequence of irregular meshes

	$\|\hat{q}-\hat{q}_h\|$		$\|\hat{u}-\hat{u}_h\|$		$\|\hat{z}-\hat{z}_h\|$	
# cells	error	rate	error	rate	error	rate
16	3.18e-02	-	1.28e-02	-	4.70e-03	-
64	1.95e-02	0.70	5.14e-03	1.32	1.28e-03	1.88
256	1.16e-02	0.75	2.07e-03	1.31	3.40e-04	1.91
1 024	6.21e-03	0.90	8.45e-04	1.29	8.94e-05	1.93
4 096	3.66e-03	0.76	3.31e-04	1.35	2.27e-05	1.98
16 384	2.21e-03	0.73	1.39e-04	1.26	5.79e-06	1.97
65 536	1.30e-03	0.77	5.69e-05	1.29	1.42e-06	2.03
expected		0.67		1.17		1.67

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