Error Control for Model Reduction by Discretization in Optimal Control of Flows

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Abstract

We present a new approach to mesh adaptivity in the numerical solution of optimal control problems governed by the stationary Navier-Stokes equations. Using the Lagrangian formalism the optimization problem is reformulated as a boundary value problem that is discretized by a Galerkin finite element method. The discretization is controlled by residual-based *a posteriori* error estimates derived by global duality arguments. This general approach facilitates control of the error with respect to any quantity of physical interest. In discretizing an optimal control problem it seems natural to control the error with respect to the given cost functional. In this way, the computed solution can directly be used in weighting the cell residuals in the *a posteriori* error estimate.

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1 Introduction

The subject of this paper is 'error control' and 'mesh design' in solving optimization problems governed by the 'incompressible' Navier-Stokes equations. The presented results are collected from [1], [2], [3], [5] and [16]. Let be given some 'cost functional' that is to be minimized on the set of 'admissible' flow fields:

$$\begin{aligned} &\text{Cost functional:} \quad && J(u,q) \;\to\; min! \\ &\text{State equation :} \quad && A(u) + Bq = f. \end{aligned} \tag{1.1}$$

The difficulty in this problem is that solving the state equation is very expensive such that the use of an economical discretization is indispensable. This naturally leads to the question of how to organize the discretization, that is the reduction of the infinite dimensional model to a finite dimensional one. This requires some preliminary thoughts:

The notion of an 'admissible' state obtained by the solution operator $q \to u =: S(q)$ needs specification. Discretization necessarily introduces a perturbation of S, that is $u_h = S_h(q_h)$ replaces $u = S(q)$. The question is which degree of 'admissibility' is needed for the optimization process. Because of limited computing resources, accuracy in solving PDEs by discretization is expensive. Therefore measures for 'admissibility' are to be chosen in accordance to the given problem. In optimal control of ODEs one naturally requires the max-norm $\|u - u_h\|_\infty$ to be small. In the context of PDEs the appropriate measure for error control is less clear. Examples are the natural 'energy' norm $\|\nabla(u - u_h)\|_2$, some weighted L^2-norm $\|\omega(u - u_h)\|_2$, or the max-norm $\|u - u_h\|_{\infty;\Omega'}$ over a subdomain $\Omega' \subset \Omega$, etc.

As an illustrative example, we consider the flow around the cross section of a circular cylinder in a channel described by the classical Navier-Stokes equations for viscous, incompressible Newtonian fluid flow (see Fig. 1):

$$-\nu\Delta v + v\cdot\nabla v + \nabla p = f, \quad \nabla\cdot v = 0, \quad \text{in } \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is the bounded flow domain. The unknowns are the velocity v and the pressure p; ν is the normalized viscosity (density $\rho \equiv 1$) and the volume force is assumed as $f \equiv 0$. At the boundary $\partial\Omega$, the usual non-slip condition is imposed along rigid parts together with suitable inflow and free-stream outflow conditions,

$$v|_{\Gamma_{rigid}} = 0, \quad v|_{\Gamma_{in}} = v^{in}, \quad \nu\partial_n v - pn|_{\Gamma_{out}} = 0. \tag{1.3}$$

The goal is to compute the drag coefficient c_{drag} of the cylinder that is defined by

$$c_{drag} = \frac{2}{\bar{U}^2 D} \int_S n{\cdot}\sigma(v,p)\psi \, ds, \tag{1.4}$$

where S is the surface of the cylinder, D its diameter, $\bar{U}$ the reference velocity, $\psi := (0,1)^T$, and $\sigma(v,p) = \frac{1}{2}\nu(\nabla v + \nabla v^T) + pI$ the stress acting on the cylinder. The Reynolds number is $Re = \bar{U}^2 D/\nu = 20$, such that the flow is stationary.

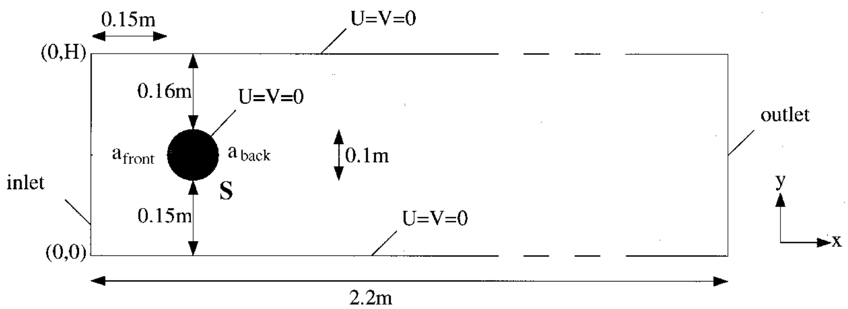

Figure 1: Geometry of the benchmark 'flow around a cylinder'

This problem is part of a flow benchmark described in Schäfer & Turek [20]. From [4], we recall some computational results obtained by a Galerkin finite element method (see Table 1).

Type of Mesh	Error in c_{drag}	# Cells	Work Units
uniform	$\sim 1\%$	500,000	300
hand-adapted	$\sim 1\%$	75,000	50
semi self-adapted	$\sim 1\%$	25,000	15
fully self-adapted	$\sim 1\%$	1,500	1

Table 1: Results for the benchmark 'flow around a cylinder' obtained on different types of meshes.

We observe that the efficiency of the computation highly depends on the design of the computational mesh. Since optimization in such models usually requires several solutions of the 'forward' problem, it is clear that economical meshes are needed that are tailored to the particular needs of the computation. Methods for constructing such 'optimal' meshes have to be based on information obtained from the computed approximation $\{v_h, p_h\}$. The main subject of this paper is the systematic derivation of such *a posteriori* error estimates and their use in an feed-back process for constructing economical meshes particularly for the purpose of optimal control.

The starting point is the reformulation of the optimal control problem as a nonlinear boundary value problem by employing the Lagrangian formalism. The resulting saddle-point system of Euler-Lagrange equations is discretized by a standard Galerkin finite element method using conforming Q_1-elements on quadrilateral meshes for all unknowns. Stability of velocity-pressure coupling and convective transport is achieved by a consistent least-squares stabilization.

For *a posteriori* error estimation in this scheme, we apply the general duality approach developed in [6] and [7] for Galerkin finite element discretizations of partial differential equations (see also the survey articles [19] and [8]). This method derives *a posteriori* estimates for the error with respect to arbitrary functional output by employing the solution of a 'dual problem'. In these estimates local cell residuals of the computed solution are weighted by certain derivatives of the 'dual solution' (generalized Green's function) that describe the dependence of the error on variations of the local mesh size. In general these weights have to be approximated by numerically solving the corresponding dual problem. This results in a feed-back process by which successively more and more accurate error bounds and economical meshes are generated. In applying this approach to optimization problems, it seems natural to base the error control on the given cost functional. In this particular case the corresponding dual solution is given in terms of the state variable, Lagrangian multiplier and control variable. Hence, the evaluation of the *a posteriori* error estimates does not require extra work for solving a dual problem and *a posteriori* error estimation is almost for free. This observation leads to a particularly simple and efficient strategy for mesh adaptation in the finite element discretization of optimal control problems.

2 Residual-based error control in flow computation

At first, we want to illustrate the common strategies for *a posteriori* error estimation and the resulting automatic mesh adaptation by a flow example similar to the benchmark discussed in the introduction. Again, the goal of the computation is the drag coefficient c_{drag}. The configuration shown in Fig. 2 has the special feature that the accuracy in computing the drag that 'lives' in the neighborhood of the cylinder surface S may be strongly effected by the presence of the 'corner singularity' at the narrowed outlet Γ_{out}. The quantitative strength of this upwind 'pollution effect' cannot be determined analytically. It has to be taken into account by the mesh design, but with an appropriate balance between discretization accuracy and solution efficiency. This balance has to be guided by computational means as will become clear by the results presented below in Fig. 4 and 5.

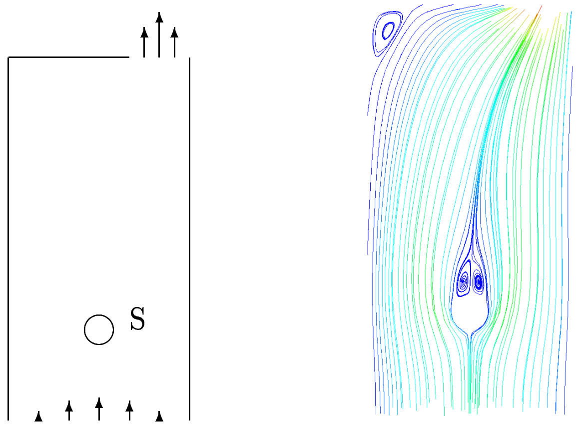

Figure 2: Configuration of the test problem and streamline plot of the velocity field for $Re = 40$.

2.1 Galerkin finite element discretization

The starting point is the usual variational formulation of the Navier-Stokes problem (1.2), (1.3). We introduce the notation $L := L^2(\Omega)$, $\hat{H} := H^1(\Omega)^2$, and $H := \{v \in H^1(\Omega)^2,\ v_{|\Gamma_{\rm in} \cup \Gamma_{\rm rigid}} = 0\}$, and set

$$\hat{V} := \hat{H} \times L, \quad V := H \times L \subset \hat{V}.$$

For pairs $u = \{v, p\},\ \varphi = \{w, q\} \in \hat{V}$, we define the semi-linear form

$$a(u; \varphi) := \nu(\nabla v, \nabla w) + (v \cdot \nabla v, w) - (p, \nabla \cdot w) + (q, \nabla \cdot v)$$

Then, with a solenoidal extension $\hat{v} \in \hat{V}$ of the inflow data v^{in}, we seek $u = \{v, p\} \in V + \{\hat{v}, 0\}$, such that

$$a(u; \varphi) = (f, w) \quad \forall \varphi = \{w, q\} \in V. \tag{2.1}$$

We assume that this problem possesses a (locally) unique solution which is stable, that is the Fréchet derivative $A'(u; \cdot, \cdot)$ is coercive.

For discretizing problem (2.1), we use a finite element method based on the quadrilateral Q_1/Q_1-Stokes element with globally continuous (isoparametric) piecewise bilinear shape functions for both unknowns, pressure and velocity, defined on regular decompositions $\mathbb{T}_h = \{K\}$ of $\bar{\Omega}$ into quadrilaterals ('cells') K. In order to ease local mesh refinement, we allow 'hanging' nodes (see Fig. 3), where the corresponding 'irregular' nodal values are eliminated from the system by linear interpolation of neighboring regular nodal values. The corresponding finite element subspaces are denoted by

$$L_h \subset L, \quad \hat{H}_h \subset \hat{H}, \quad H_h \subset H, \quad \hat{V}_h := L_h \times \hat{H}_h, \quad V_h := L_h \times H_h,$$

and $\hat{v}_h \in \hat{H}_h$ is a suitable interpolation of the boundary function $\hat{v}$. This construction is oriented by the situation of a polygonal domain Ω for which the boundary $\partial\Omega$ is exactly matched by the mesh domain $\Omega_h := \cup\{K \in \mathbb{T}_h\}$. In the case of a general curved boundary (as in the flow example considered) some standard modifications are necessary which are omitted (see, e.g., Girault & Raviart [13] and Brenner & Scott [10]).

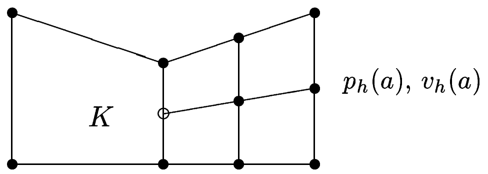

Figure 3: Quadrilateral mesh patch for the Q_1/Q_1-Stokes element with a 'hanging node'

In order to obtain a stable discretization of (2.1) in these spaces with 'equal-order interpolation' of pressure and velocity, we use the least-squares technique proposed by Hughes, Franca & Balestra [15]. Following Hughes & Brooks [14], a similar approach is employed for stabilizing the convection term (see [9] or [3]). The Navier-Stokes system can be written in vector form for the unknown $u = \{v, p\} \in \hat{V}$ as

$$A(u) := \begin{bmatrix} -\nu\Delta v + v\cdot\nabla v + \nabla p \\ \nabla\cdot v \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} =: F.$$

Then, the strong solution $u = \{v, p\}$ of (2.1) satisfies $A(u) = F$ in the L^2-sense. To the operator $A(\cdot)$, we associate a derivative $A'(u)$ at u and an approximation $S(u)$ which act on $\varphi = \{q, w\} \in \hat{V}$ like

$$A'(u)\varphi := \begin{bmatrix} -\nu\Delta w + v\cdot\nabla w + w\cdot\nabla v + \nabla q \\ \nabla\cdot w \end{bmatrix}, \qquad S(u)\varphi := \begin{bmatrix} v\cdot\nabla w + \nabla q \\ 0 \end{bmatrix}.$$

With this notation, we introduce the stabilized form

$$a_h(u;\varphi) := a(u;\varphi) + (A(u) - F, S(u)\varphi)_h,$$

with the mesh-dependent inner product and norm

$$(v,w)_h := \sum_{K\in\mathbb{T}_h} \delta_K\,(v,w)_K, \quad \|v\|_h = (v,v)_h^{1/2}.$$

The stabilization parameter is chosen according to

$$\delta_K = \alpha\big(\nu h_K^{-2}, \beta\,|v_h|_{K;\infty} h_K^{-1}\big)^{-1}, \quad \delta := \max_{K\in\mathbb{T}_h} \delta_K, \tag{2.2}$$

with the heuristic values $\alpha = \frac{1}{12}$, $\beta = \frac{1}{6}$. Now, in the discrete problems, we seek $\{v_h, p_h\} \in V_h + \hat{u}_h$, such that

$$a_h(u_h;\varphi_h) = f(\varphi_h) \quad \forall \varphi_h \in V_h. \tag{2.3}$$

This approximation is fully consistent in the sense that the solution $u = \{p, v\}$ also satisfies (2.3). This implies 'Galerkin orthogonality' for the error $e = \{e^v, e^p\} := \{v - v_h, p - p_h\}$ with respect to the form $a_h(\cdot;\cdot)$ in the sense

$$a_h(u;\varphi_h) - a_h(u_h;\varphi_h) = 0, \quad \varphi_h \in V_h. \tag{2.4}$$

2.2 Adaptive mesh control

We now turn to the question of *a posteriori* error estimation and mesh construction in the scheme (2.3). Usually, automatic mesh adaptation is guided by 'cell error indicators' η_K which are obtained from the computed solution $\{v_h, p_h\}$. Examples of common indicators are:

- *Vorticity indicator*: $\eta_K := \|\nabla \times v_h\|_K$.
- *Pressure-gradient indicator*: $\eta_K := \|\nabla p_h\|_K$.
- *'Energy-norm error' indicator*:

  $$\eta_K := \|R^p(u_h)\|_K + \|R^v(u_h)\|_K + h_K^{1/2}\|r^v(u_h)\|_{\partial K},$$

  with the cell and edge residuals defined by

  $$\begin{aligned} R^p(u_h)_{|K} &:= \nabla\cdot v_h\,, \\ R^v(u_h)_{|K} &:= -\nu\Delta v_h + v_h\cdot\nabla v_h + \nabla p_h\,, \\ r^v(u_h)_{|\Gamma} &:= \begin{cases} -\frac{1}{2}[\nu\partial_n v_h - p_h n], & \text{if } \Gamma \not\subset \partial\Omega, \\ -(\nu\partial_n v_h - p_h n), & \text{if } \Gamma \subset \Gamma_{out}, \\ 0, & \text{if } \Gamma \subset \Gamma_{rigid} \cup \Gamma_{in}, \end{cases} \end{aligned}$$

  where [...] indicates the jump across an edge $\Gamma \subset \partial K \setminus \partial\Omega$.

The vorticity and the pressure-gradient indicators measure the 'smoothness' of the computed solution $\{v_h, p_h\}$ while the 'energy-error' indicator additionally contains information about local conservation of mass and momentum. However, neither of them contains information about the effect of changing the local mesh size on the error in the target quantity c_{drag}. This is only achieved by the 'weighted' indicators derived from a new *a posteriori* error estimator $\eta_\omega(u_h)$ that will be derived below. In Fig. 4 and 5, we compare the results for computing the drag coefficient c_{drag} on meshes which are constructed by using the different error indicators with that obtained on uniformly refined meshes.

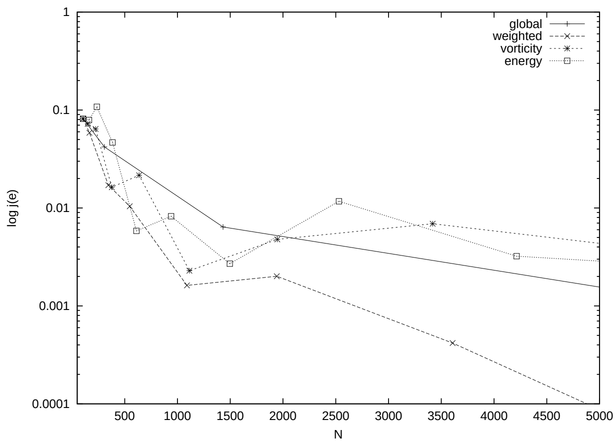

Figure 4: Mesh efficiency obtained by uniform refinement ('global' $+$), the weighted indicator ('weighted' $\times$), the vorticity indicator ('vorticity' $*$), and the energy indicator ('energy' $\square$).

Remark 1. *In our computation the drag coefficient has actually been evaluated from the formula*

$$c_{drag} = \frac{2}{\hat{H}^2 D} \int_\Omega \{\sigma(p,v)\nabla\bar{\psi} + \nabla\cdot\sigma(p,v)\bar{\psi}\}\,dx\,,$$

where $\bar{\psi}$ *is an extension of the directional vector* $\psi := (0,1)^T$ *from* S *to* Ω *with support along* S. *Since* $\{v,p\}$ *satisfies the Navier-Stokes equations, and setting* $\bar{\varphi} := \{\bar{\psi}, 0\}$, *it follows that*

$$c_{drag} = a(u,\bar{\varphi}) - f(\bar{\varphi}). \tag{2.5}$$

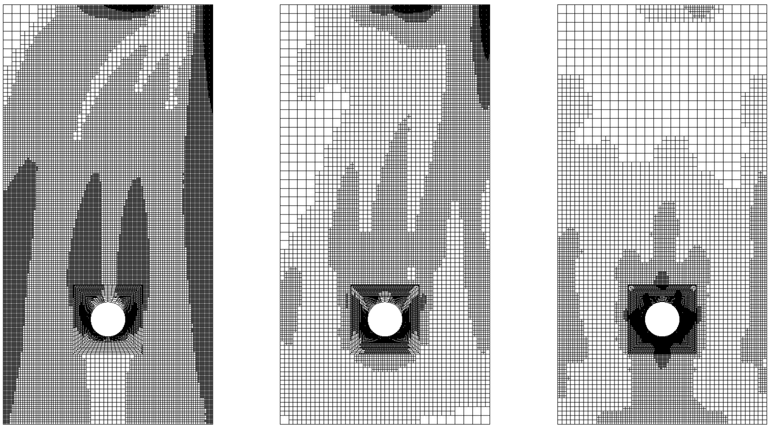

Figure 5: Meshes with about 5000 cells obtained by the vorticity indicator (left), the 'energy' indicator (middle), and the weighted indicator (right).

By integration by parts, one sees that this definition is independent of the choice of $\bar{\psi}$ *and that it is equivalent to the original one (1.4). However, on the discrete level the two formulations differ. In fact, computational experience shows that formula (2.5) yields significantly more accurate approximations of the drag coefficient (see Giles, Larsson, Levenstam & Süli [12] and Becker [1]).*

The results presented for this test problem motivate the development of a more systematic method for *a posteriori* error estimation with emphasize on the error in certain local output quantities of the solution. This approach, called 'DWR method' ('Dual-Weighted-Residual' method), will be introduced in the following subsection (see [1], [6] or [8] for a more detailed description). For other contributions to 'duality-based' error estimation see Eriksson, Estep, Hanspo & Johnson [11] and Machiels, Patera & Peraire [17]. References to the classical 'energy-norm' based *a posteriori* error analysis for the Navier-Stokes equations are, e.g., Verfürth [21] and Oden, Weihan & Ainsworth [18],

2.3 The DWR method for a posteriori error estimation

We continue using the notation from above. In order to estimate the error in the drag c_{drag} or any other quantity $J(u)$ of physical interest such as point values of pressure ('pressure drop'), averages of vorticity localized to certain recirculation zones, or line averages of tracer concentrations corresponding to laser measurements, we use a 'duality argument'. We consider the following h-dependent 'dual problem'

$$a_h'(u;\varphi,z) := a'(u;\varphi,z) + (S(u)\varphi, S(u)z)_h = J(\varphi) \quad \forall \varphi \in V, \tag{2.6}$$

with the approximate derivative $S(u)$ defined above. Since the stabilizing bilinear form $(S(u)\cdot, S(u)\cdot)_h$ is coercive, the strong coercivity of $a'(u;\cdot,\cdot)$ implies also the (unique) solvability of the adjoint problem (2.6). With this notation, we have the following result.

Proposition 1. *For a (linear) functional $J(\cdot)$ let $z=\{z^v,z^p\}\in V$ be the solution of the linearized dual problem (2.6). Then, we have the a posteriori error estimate*

$$|J(e)| \;\le\; \eta_\omega(u_h) \;:=\; \sum_{K\in\mathbb{T}_h} \{\rho_K^v\omega_K^v + \rho_K^p\omega_K^p + ...\} + R_h\,, \tag{2.7}$$

where the residual terms and weights are given by

$$\begin{aligned} \rho_K^v \;&:= \|R^v(u_h)\|_K + h_K^{-1/2}\|r^v(u_h)\|_{\partial K}, \quad \rho_K^p := \|R^p(u_h)\|_K\,,\\ \omega_K^v \;&= \|z^v - i_h z^v\|_K + h_K^{1/2}\|z^v - i_h z^v\|_{\partial K}, \quad \omega_K^p := \|z^p - i_h z^p\|_K\,, \end{aligned}$$

with local approximations $\{i_h z^v, i_h z^p\}\in V_h$ to $\{z^v,z^p\}$. The dots '...' in (2.7) stand for additional terms measuring the effect of the stabilization and the errors in approximating the inflow data and the curved cylinder boundary. The remainder term is essentially estimated by

$$|R_h| \;\le\; \|e^v\|\,\|\nabla e^v\|\,\|z^v\|_\infty. \tag{2.8}$$

Proof. (i) For the proof, we recall an abstract result from [8] (see also [6], [1] and [3]) on *a posteriori* error representations for Galerkin approximations of the form considered. For simplicity, we assume the inflow data v^{in} to be exactly matched by the finite element approximation, that is $e^v \in V$. Then, setting $\varphi = e$ in the dual equation (2.6), we obtain

$$J(e) \;=\; a_h'(u;e,z) \;=\; a'(u;e,z) + (S(u)e, S(u)z)_h.$$

Further, noting that

$$a_h(u;z) = a_h(u_h;z) + a_h'(u_h;e,z) + \int_0^1 a_h''(u_h + se;e,e,z)\,(1-s)\,ds,$$

and $a_h(u;z) = f(z)$, we conclude

$$J(e) = f(z) - a_h(u_h;z) + R_h(u,u_h;e,e,z),$$

with the remainder term

$$R_h(u,u_h;e,e,z) := \int_0^1 a_h''(u_h+se;e,e,z)\,(1-s)\,ds.$$

Then, using that $a_h(u_h;\psi_h)=0$ for $\psi_h\in V_h$, we obtain

$$J(e) = f(z-\psi_h) - a_h(u_h;z-\psi_h) + R_h(u,u_h;e,e,z), \tag{2.9}$$

for an arbitrary $\psi_h\in V_h$.
(ii) Next, we have to evaluate the residual $\rho(u_h;\psi)=f(\psi)-a_h(u_h;\psi)$ for $\psi=\{w,q\}\in V$. Neglecting the errors due to the approximation of the inflow data $\hat v$ and the curved cylinder boundary S, there holds

$$\begin{aligned}\rho(u_h;\varphi) \;&= (f,w) - \nu(\nabla v_h,\nabla w) - (v_h\cdot\nabla v_h, w) + (p_h,\nabla\cdot w)\\ &\quad - (q,\nabla\cdot u_h) - (A(u_h)-F, S(u_h)\varphi)_h.\end{aligned}$$

Splitting the integrals into their contributions from each cell $K\in\mathbb{T}_h$ and integrating cell-wise by parts yields

$$\begin{aligned}\rho(u_h;\varphi) \;&=\; \sum_{K\in\mathbb{T}_h}\Big\{(R^v(u_h),w)_K + (r^v(u_h),w)_{\partial K} + (q,R^p(u_h))_K\\ &\qquad + \delta_K(R^v(u_h), v_h\cdot\nabla w+\nabla q)_K\Big\}.\end{aligned}$$

From this, we obtain the error estimate (2.7) if we set $\varphi := z-z_h$. It remains to estimate the remainder term R_h which, observing that $A(u)-F=0$, in the present situation has the form

$$R_h = (S(u)e,S(u)z)_h) - (A'(u)e,S(u)z)_h + \int_0^1 a_h''(u_h+se;e,e,z)s\,ds.$$

We recall that for arguments $u=\{v,p\},\varphi=\{w,q\}$ and $z=\{z^v,z^p\}$,

$$A'(u)\varphi := \begin{bmatrix}-\nu\Delta w + v\cdot\nabla w + w\cdot\nabla v + \nabla q\\ \nabla\cdot w\end{bmatrix},\quad S(u)\varphi := \begin{bmatrix} v\cdot\nabla w+\nabla q\\ 0\end{bmatrix},$$
$$A''(u)\varphi z := \begin{bmatrix} w\cdot\nabla z^v + z^v\cdot\nabla w\\ 0\end{bmatrix},\quad S'(u)\varphi z := \begin{bmatrix} w\cdot\nabla z^v\\ 0\end{bmatrix}.$$

Hence, the first two terms in the remainder R_h can be written in the form

$$(S(u)e,S(u)z)_h - (A'(u)e,S(u)z)_h = -\,(-\nu\Delta e^v + e^v\cdot\nabla v, v\cdot\nabla z^v + \nabla z^p)_h.$$

Further, for the arguments $u=\{v,p\}$, $\varphi=\{w,q\}\in V$ and $z=\{z^v,z^p\}\in V$, the first derivative of $a_h(\cdot;\cdot)$ has the explicit form

$$a_h'(u;\varphi,z) \;=\; (A'(u)\varphi,z)+(A'(u)\varphi,S(u)z)_h+(A(u),S'(u)\varphi z)_h,$$

with $a'(u)\varphi$ and $S(u)z$ defined as above. Since the nonlinearity of $a(\cdot;\cdot)$ is quadratic, and $S''(u)\equiv 0$, for arguments $u_s=\{v_s,p_s\}$, $e=\{e^v,e^p\}$ and $z=\{z^v,z^p\}$, the second derivative of $a_h(\cdot;\cdot)$ has the form

$$\begin{aligned} a_h''(u_s;e,e,z) = {} & 2(e^v\cdot\nabla e^v,z^v)+2(e^v\cdot\nabla e^v,v_s\cdot\nabla z^v+\nabla z^p)_h \\ & +2\,(-\nu\Delta e^v+v_s\cdot\nabla e^v+e^v\cdot\nabla v_s+\nabla e^p,e^v\cdot\nabla z^v)_h\,. \end{aligned}$$

From this, we easily infer the proposed bound for the remainder term. □

In the estimate (2.7) the additional terms representing the errors related to the stabilization and the errors in approximating the inflow data and the curved boundary component S are neglected. They are expected to be small compared to the other residual terms. The practical use of the error estimate (2.7) will be discussed in the next section.

2.4 Practical mesh adaptation

The approximation of equation (1.2) by a Galerkin method has been described in a functional analytic setting. For solving it on a computer, we have to convert the *discrete* problem (2.1) into an algebraic equation. To this end, we choose a basis $\{\varphi^\nu,\,\nu=1,...,N=\dim V_h\}$ of the subspace $V_h\subset V$ (for instance the standard 'nodal function basis' in a finite element scheme) and seek for a solution in the form

$$u_h=\sum\nolimits_{\nu=1}^{N} x_\nu\varphi^\nu+\hat{u}_h\,.$$

Inserting this into equation (2.1) results in a nonlinear algebraic system

$$a(u_h;\varphi^\nu)=f(\varphi^\nu),\quad \nu=1,...,N, \tag{2.10}$$

for the unknown coefficient vector $x=\{x_\nu\}_{\nu=1}^N$. This system may be solved for instance by a Newton iteration. In the context of adaptive discretization, this solution process is usually coupled with successive mesh refinement. The resulting *nested* algorithm reads as follows.

Let a desired error tolerance TOL or a maximum mesh complexity $N_{\max}$ be given. Starting from a coarse initial mesh $\mathbb{T}_0$, a hierarchy of successively refined meshes $\mathbb{T}_i,\ i\ge 1$, and corresponding finite element spaces $V_i\subset \hat V_i$ is generated as follows:

(0) *Initialization* $i=0:$ Compute an initial approximation $u_0\in\hat V_0$.

(i) *Defect correction iteration:* For $i\ge 1$, start with $u_i^{(0)}:=u_{i-1}\in\hat V_i$.

(ii) *Iteration step:* For $j\ge 0$ evaluate the defect

$$(d_i^{(j)},\varphi_i):=f(\varphi_i)-a(u_i^{(j)};\varphi_i),\quad \varphi_i\in V_i. \tag{2.11}$$

Choose a suitable approximation $\tilde a'(u_i^{(j)};\cdot,\cdot)$ to $a'(u_i^{(j)};\cdot,\cdot)$ (with good stability and solution properties) and compute a correction $v_i^{(j)}\in V_i$ from the linear equation

$$\tilde a'(u_i^{(j)};v_i^{(j)},\varphi_i)=(d_i^{(j)},\varphi_i)\quad\forall\varphi_i\in V_i. \tag{2.12}$$

For this, Krylov-space and multigrid methods are employed using the hierarchy of meshes $\{\mathbb{T}_i,...,\mathbb{T}_0\}$ already constructed. Then, update $u_i^{(j+1)}=u_i^{(j)}+\lambda_i v_i^{(j)}$, with some relaxation parameter $\lambda_i\in(0,1]$, set $j:=j+1$ and go back to (2). This process is repeated until a limit $\tilde u_i\in\hat V_i$, is reached with a certain prescribed accuracy.

(iii) *Error estimation:* Accept $u_i:=\tilde u_i$ as the solution on mesh $\mathbb{T}_i$ and solve the discrete linearized dual problem

$$a'(u_i;\varphi_i,z_i)=J(\varphi_i)\quad\forall\varphi_i\in V_i, \tag{2.13}$$

for $z_i\in V_i$, and evaluate the *a posteriori* error estimate

$$|J(e_i)|\approx\eta(u_i):=\sum\nolimits_{K\in\mathbb{T}_i}\eta_K, \tag{2.14}$$

with the cell error indicators $\eta_K:=\rho_K^v\omega_K^v+\rho_K^p\omega_K^p$ defined in Proposition 1. The weights ω_K^v and ω_K^p are computed from the discrete dual solution $\{z_i^v,z_i^p\}\in V_i$ by patchwise higher-order (bi-quadratic) interpolation $\{i_{2h}^{(2)}z_i^v,i_{2h}^{(2)}z_i^p\}$ and setting

$$\omega_K^v\approx\|i_{2h}^{(2)}z_i^v-z_i^v\|_K+h_K^{1/2}\|i_{2h}^{(2)}z_i^v-z_i^v\|_{\partial K},\quad \omega_K^p\approx\|i_{2h}^{(2)}z_i^p-z_i^p\|_K\,.$$

Then, the adaptation of the current mesh $\mathbb{T}_i$ is organized according to one of the following strategies.

(a) 'Error-Balancing Strategy': One cycles through the mesh and equilibrates the local error indicators by $\eta_K\approx \mathrm{TOL}/N$ with $N:=\#\{K\in\mathbb{T}_h\}$. This leads to $\eta\approx TOL$, but requires iteration with respect to N.

(b) 'Fixed-Fraction Strategy': One orders the cells according to the size of η_K and refines a certain percentage (say 20%) of cells with largest η_K (or those which make up 20% of the estimator value η) and coarsen those cells with smallest η_K. By this strategy, one may achieve a prescribed rate of increase of N.

For controlling the reliability of the error bound (2.14), one may check whether the change $\|z_i - z_{i-1}\|$ is sufficiently small; if this is not the case, additional global mesh refinement is advisable. If $\eta(u_i) \le TOL$ or $N_i \ge N_{\max}$, then stop. Otherwise, cellwise mesh adaptation yields the new mesh $\mathbb{T}_{i+1}$. Then, set $i := i+1$, and go back to (i).

We note that the evaluation of the *a posteriori* error estimate (2.14) involves only the solution of *linearized* problems. Hence, the whole error estimation may amount only to a relatively small fraction of the total cost for the solution process. This has to be compared to the usually much higher cost of working on non-optimized meshes.

3 Principles of optimization

Now, we turn back to optimization. There are two main approaches to solving the general optimal control problem (1.1). Let $S : q \to \{v, p\}$ denote the solution operator of the state equation.
i) Direct method:

$$J(u) = J(S^{-1}(q)) \;\to\; min.$$

For solving this problem, one may use one of the usual descent methods (gradient, Newton, CG, etc.). The problem of this approach is the efficient computation of the derivatives $\partial_q J(S^{-1}(q))$ via difference quotients which requires the frequent solution of the state equation.
ii) Indirect method: The Lagrangian approach introduces the Lagrangian functional

$$L(u,q,\lambda) := J(u) + a(u;\lambda) - f(\lambda) + \tfrac{1}{2}\alpha\, r(u,q),$$

where $\alpha > 0$ is a regularization parameter. Then, possible extrema of $J(\cdot)$, are given as stationary points of $L(\cdot)$, that is as solutions of the Euler-Lagrange system

$$\nabla_{u,q,\lambda} L(u,q,\lambda) = 0.$$

This approach has the advantage that the exact inversion of S is not required. This may lead to an optimization algorithm that costs only a moderate multiple of one forward solution of the state equation.

3.1 Abstract a posteriori error estimation

Let V and Q be two Hilbert spaces with scalar products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_Q$, respectively. Further, let $J(\cdot)$ be a functional, $a(\cdot;\cdot)$ a semi-linear form on $V\times V$, and $b(\cdot,\cdot)$ a bilinear form on $Q\times V$. The functional $J(\cdot)$ and the form $a(\cdot;\cdot)$ are supposed to posses up to third-order directional derivatives denoted by $J'(\cdot;\cdot)$, ..., $J'''(\cdot;\cdot,\cdot,\cdot)$, and $a'(\cdot;\cdot,\cdot)$, ..., $a'''(\cdot;\cdot,\cdot,\cdot,\cdot)$, respectively. With this notation, we consider the abstract optimization problem

$$J_\alpha(u,q) := J(u) + \tfrac{1}{2}\alpha\|q\|^2 \;\to\; \min!, \tag{3.1}$$

under the equation constraint

$$a(u;\varphi) + b(q;\varphi) = (f,\varphi) \quad \forall\varphi\in V. \tag{3.2}$$

For solving this problem, we use the classical Lagrangian approach. Introducing the Lagrangian

$$L(u,q,\lambda) \;:=\; J_\alpha(u,q) + a(u;\lambda) + b(q,\lambda) - (f,\lambda),$$

which is defined on the product space $X := V\times Q\times V$, possible extrema are among the stationary points $x=\{u,q,\lambda\}\in X$ of L. Hence, it is our goal to approximate solutions of the variational equation

$$L'(x;y) = 0 \quad \forall y\in X., \tag{3.3}$$

where $L'(x;y)$ denotes the derivative of $L(\cdot)$ at x in direction y. Equation (3.3) is equivalent to the following saddle-point system:

$$\begin{aligned} J'(u;\varphi) + a'(u;\varphi,\lambda) \;&=\; 0 \qquad \forall\varphi\in V,\\ a(u;\psi) + b(q,\psi) \;&=\; (f,\psi) \qquad \forall\psi\in V,\\ b(\chi,\lambda) + \alpha(\chi,q) \;&=\; 0 \qquad \forall\chi\in Q. \end{aligned} \tag{3.4}$$

Equation (3.3) is approximated by a Galerkin method. To this end, we introduce finite dimensional subspace $V_h\subset V$ and $Q_h\subset Q$ parametrized by $h\in\mathbb{R}_+$, and set $X_h := V_h\times Q_h\times V_h \subset X$. Then, we determine approximations $x_h=\{u_h,q_h,\lambda_h\}\in X_h$ by

$$L'(x_h;y_h) = 0 \quad \forall y_h\in X_h. \tag{3.5}$$

Accordingly, this equation is equivalent to the discrete saddle-point problem

$$\begin{aligned} J'(u_h;\varphi_h) + a'(u_h;\varphi_h,\lambda_h) \;&=\; 0 \qquad \forall\varphi_h\in V,\\ a(u_h;\psi_h) + b(q_h,\psi_h) \;&=\; (f,\psi_h) \qquad \forall\psi_h\in V_h,\\ b(\chi_h,\lambda_h) + \alpha(\chi_h,q_h) \;&=\; 0 \qquad \forall\chi_h\in Q_h. \end{aligned} \tag{3.6}$$

The following proposition states an *a posteriori* estimate for the error $e := x - x_h$ in terms of the 'residual' $L'(x_h, \cdot)$.

Proposition 2. *We have the error representation*

$$J_\alpha(u,q) - J_\alpha(u_h,q_h) = \tfrac{1}{2}\, L'(x_h; x - y_h) + R_h, \tag{3.7}$$

with an arbitrary $y_h \in X_h$ *, and a remainder* R_h *given by*

$$R_h = \tfrac{1}{2} \int_0^1 L'''(x_h + te; e, e, e)\, s(1-s)\, ds.$$

Proof. The proof is by arguments from elementary calculus. Since u and u_h satisfy (3.2) and (3.6), respectively, we have

$$\begin{aligned} L(x) - L(x_h) &= J_\alpha(u,q) - J_\alpha(u_h,q_h) \\ &\quad + a(u;\lambda) + b(q,\lambda) - a(u_h;\lambda_h) - b(q_h,\lambda_h) \\ &= J_\alpha(u,q) - J_\alpha(u_h,q_h). \end{aligned}$$

We note the identity

$$L(x) - L(x_h) = L'(x_h; e) + L''(x_h; e.e) - \int_0^1 L'''(x_h + se; e, e, e)\, (s-1)^2\, ds.$$

Further, the relation

$$0 = L'(x; e) = L'(x_h; e) + L''(x_h; e, e) - \int_0^1 L'''(x_h + se; e, e, e)\, (s-1)\, ds$$

implies that

$$L''(x_h; e, e) = \; - L'(x_h; e) + \int_0^1 L'''(x_h + se; e, e, e)\, (s-1)\, ds.$$

Since $(s-1)^2 + s - 1 = s(s-1)$, it follows that

$$L(x) - L(x_h) = \tfrac{1}{2} L'(x_h; e) + \int_0^1 L'''(x_h + se; e, e, e)\, s(s-1)\, ds$$

Then, using $L'(x_h; y_h)$ for $y_h \in X_h$, the proof is completed. □

3.2 The adaptive optimization process

We outline the main steps in the adaptive solution process for the optimization problem (3.1) and (3.2).

Step 1. Form the first-order necessary condition

$$L'(x;y) = 0 \quad \forall y \in X,$$

and discretize this by a finite element Galerkin scheme

$$L'(x_h;y_h) = 0 \quad \forall y_h \in X_h.$$

Step 2. Evaluate the residual $\frac{1}{2}L'(x_h; x - y_h)$ by using a suitable approximation $\tilde{x}_h$ to x, and neglect the remainder term R_h to obtain

$$J_\alpha(u,q) - J_\alpha(x_h,q_h)) \approx \tfrac{1}{2}L'(x_h, \tilde{x}_h - y_h).$$

Step 3. Adapt the mesh by using the cell-wise information contained in the residual:

$$|J_\alpha(u,q) - J_\alpha(x_h,q_h))| \approx \sum_{K \in \mathbb{T}_h} \eta_K(x_h),$$

yielding 'optimal' control q_h^{opt}, state $u_h^{opt} = S_h^{-1}(q_h^{opt})$, and corresponding cost $J_\alpha(u_h^{opt}, q_h^{opt})$.

Step 4. Compute a more accurate 'admissible' state $\tilde{u}_h^{opt}$ be solving the state equation (3.2) with the optimized control q_h^{opt} on a finer mesh $\mathbb{T}_{\tilde{h}}$:

$$\tilde{u}_h^{opt} = S_{\tilde{h}}^{-1}(q_h^{opt}).$$

4 Application to drag minimization

Now, we want to apply the formalism of the previous section to the particular situation of drag minimization in the Navier-Stokes equations. The configuration is shown in Fig. 6. The Reynolds number is $Re = \bar{U}^2 D/\nu = 40$. The flow is controlled by the pressure at the two openings Γ_{Q_1} and Γ_{Q_2}.

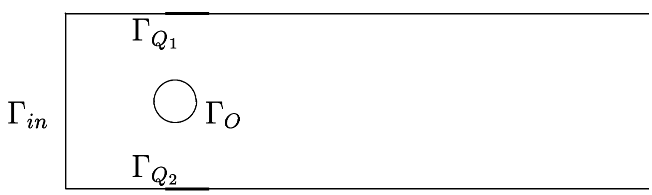

Figure 6: Configuration of the flow control problem.

4.1 A posteriori error estimates

In the present situation the cost functional is $J(u) := c_{drag}$, with the drag coefficient given in the form (2.5), that is stable on $\hat{V}$, such that no regularization is needed ($\alpha = 0$). Further, using the notation from above, the equation of state reads

$$a_h(u;\varphi) + b(q,\varphi) = (f,\psi) \quad \forall \varphi = \{\psi,\chi\} \in V, \tag{4.1}$$

with the bilinear 'control form'

$$b(q,\varphi) := -(q, n\cdot\psi)_{\Gamma_Q}.$$

The control q is chosen constant at the two components of the control boundary $\Gamma_Q = \Gamma_{Q_1} \cup \Gamma_{Q_2}$ and therefore spans a two-dimensional control space $Q = \mathbb{R}^2$. Hence, the solution space for the optimal control problem is $\hat{V} \times \mathbb{R}^2 \times V$. In this case, noting that $J'(u;\varphi) = J(\varphi)$ the abstract Euler-Lagrange system (3.4) takes the form

$$\begin{aligned} a_h'(u;\varphi,\lambda) &= -J(\varphi) \qquad \forall \varphi \in V, \\ a_h(u;\psi) + b(q,\psi) &= (f,\psi) \qquad \forall \psi \in V, \\ b(\chi,\lambda) &= 0 \qquad \forall \chi \in Q, \end{aligned} \tag{4.2}$$

for $\{u,q,\lambda\} \in \{V+\hat{v}\} \times \mathbb{R}^2 \times V$, where the 'primal' (state) variable $u = \{v,p\}$ and the 'dual' (adjoint) variable $\lambda = \{w,\pi\}$ each consist of a velocity and a pressure component. We note that in this particular case, the adjoint variable λ and the state variable u are coupled only through the nonlinearity in $a_h(\cdot;\cdot)$. The corresponding discrete system reads

$$\begin{aligned} a_h'(u_h;\varphi_h,\lambda_h) &= -J(\varphi_h) \qquad \forall \varphi_h \in V, \\ a_h(u_h;\psi_h) + b(q_h,\psi_h) &= (f,\psi_h) \qquad \forall \psi_h \in V_h, \\ b(\chi_h,\lambda_h) &= 0 \qquad \forall \chi_h \in Q_h, \end{aligned} \tag{4.3}$$

for $\{u_h,q_h,\lambda_h\} \in \{V_h+\hat{v}_h\} \times \mathbb{R}^2 \times V_h$. To this discretization, we associate the following 'primal' and 'dual' residual terms:

$$\begin{aligned} R^p(u_h)_{|K} &:= \nabla\cdot v_h, \\ R^v(u_h)_{|K} &:= -\nu\Delta v_h + v_h\cdot\nabla v_h + \nabla p_h, \\ r^v(u_h)_{|\Gamma} &:= \begin{cases} -\frac{1}{2}[\nu\partial_n v_h - p_h n], & \text{if } \Gamma \not\subset \partial\Omega, \\ -\nu\partial_n v_h + p_h n, & \text{if } \Gamma \subset \Gamma_{out}, \\ -\nu\partial_n v_h + (p_h - q)n, & \text{if } \Gamma \subset \Gamma_Q, \\ 0, & \text{if } \Gamma \subset \Gamma_{rigid} \cup \Gamma_{in}, \end{cases} \end{aligned}$$

$$
\begin{aligned}
R^p_*(\lambda_h)_{|K} &:= -\nabla\cdot w_h \\
R^v_*(\lambda_h)_{|K} &:= -\Delta w_h - v_h\cdot\nabla w_h + (\nabla v_h)^T w_h - \nabla \pi_h \\
r^v_*(\lambda_h)_{|\Gamma} &:= \begin{cases} -\frac{1}{2}[\nu\partial_n w_h + (v_h\cdot n)w_h + \pi n], & \text{if } \Gamma \not\subset \partial\Omega, \\ -\nu\partial_n w_h + (v_h\cdot n)w_h + \pi n, & \text{if } \Gamma \subset \Gamma_{out}\cup\Gamma_Q, \\ 0, \ \text{if } \Gamma \subset \Gamma_{rigid}\cup\Gamma_{in}, \end{cases}
\end{aligned}
$$

Now, applying Proposition 2 to this situation gives us the following result.

Proposition 3. *For the discretization of the drag minimization problem, we have the a posteriori error estimate*

$$
(4.4) \qquad |J(u-u_h)| \approx \sum_{K\in\mathbb{T}} h_K^2 \Big\{ \rho_K^v\,\omega_K^{v*} + \rho_K^p\,\omega_K^{p*} + \rho_K^{v*}\,\omega_K^v + \rho_K^{p*}\,\omega_K^p \Big\},
$$

with the cell residuals and weights defined by

$$
\begin{aligned}
\rho_K^v &:= \|R^v(u_h)\|_K + h_K^{-1/2}\|r^v(u_h)\|_{\partial K}, & \rho_K^p &:= \|R^p(u_h)\|_K\,, \\
\omega_K^{v*} &= \|w - i_h w\|_K + h_K^{1/2}\|w - i_h w\|_{\partial K}, & \omega_K^{p*} &:= \|\pi - i_h\pi\|_K\,, \\
\rho_K^{v*} &:= \|R^v_*(w_h)\|_K + h_K^{-1/2}\|r^v_*(w_h)\|_{\partial K}, & \rho_K^{p*} &:= \|R^p_*(\pi_h)\|_K\,, \\
\omega_K^v &= \|v - i_h v\|_K + h_K^{1/2}\|v - i_h v\|_{\partial K}, & \omega_K^p &:= \|p - i_h p\|_K\,,
\end{aligned}
$$

where $i_h u,\, i_h w \in V_h$ and $i_h\pi,\, i_h\pi \in V_h$ are suitable local approximations. The remainder term neglected in (4.4) is cubic in $e = u–u_h$ and $e^* := \lambda–\lambda_h$. Furthermore, the terms representing the error in approximating the inflow data and the curved surface S as well as all terms related to the least-squares stabilization are omitted.

The proof follows the line of argument already used in the proof of Proposition 1 and is therefore omitted (see [2] for more details). Again the residual terms in the error estimate (4.4) can be evaluated directly from the computed solutions u_h and λ_h. The weights are approximated as described in Section 2.4 by using patchwise higher order interpolation of u_h and λ_h.

4.2 Numerical results

We report some computational results from [1] for the drag minimization problem. Fig. 7 shows streamline plots of the uncontrolled ($q = 0$) primal solution and the corresponding adjoint solution. Notice that even in this case, there is some flow going through Γ_Q, since the pressure is implicitly normalized to zero at Γ_Q by the outflow boundary condition. In Fig. 8, streamline plots of the controlled flow ($q = q_{opt}$) and the corresponding adjoint flow are shown.

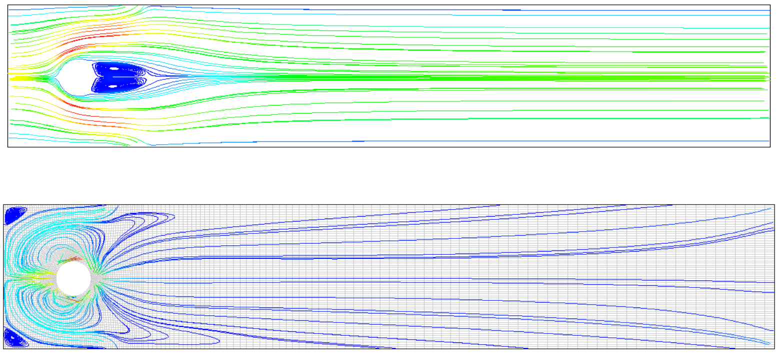

Figure 7: Uncontrolled flow ($q = 0$): primal velocity (top) and dual velocity (bottom).

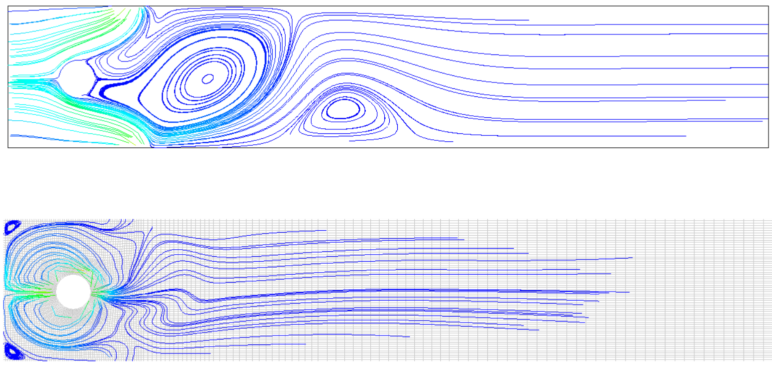

Figure 8: Controlled flow: primal velocity (top) and dual velocity (bottom).

In Table 2, we compare the computed optimal values of the drag coefficient on the meshes shown in Fig. 9 with results obtained on globally refined meshes. It is clear from these numbers that a significant reduction in the dimension of the discrete model is possible by using appropriately refined meshes. The computation required altogether 8 Newton iterations to achieve a reduction of the drag coefficient of about 40%.

Uniform refinement		Adaptive refinement	
N	c_{drag}	N	c_{drag}
10512	3.31321	1572	3.28625
41504	3.21096	4264	3.16723
164928	3.11800	11146	3.11972

Table 2: Uniform refinement (left) versus adaptive refinement (right).

Finally, Fig. 9 shows adaptively refined meshes. The locally refined meshes produced by the adaptive algorithm seem to contradict intuition since the recirculation behind the cylinder is not so well resolved. However, due to the particular structure of the optimal velocity field (most of the flow leaves the domain at the control boundary), it might be clear that this recirculation does not significantly influence the cost functional. Instead, a strong local refinement near the cylinder, where the cost functional is evaluated, as well as near the control boundary is produced.

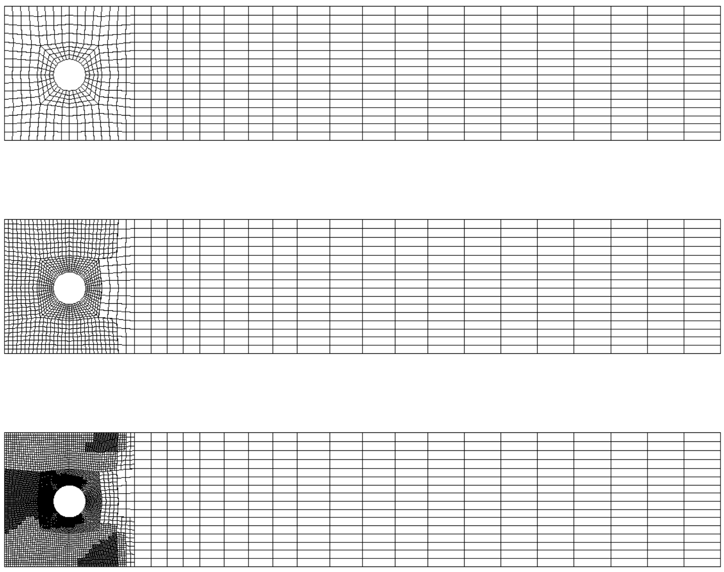

Figure 9: Automatically refined meshes for the drag-minimization problem.

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