An interpolation operator for $H^1$ functions on general quadrilateral and hexahedral meshes with hanging nodes

Vincent Heuveline * Friedhelm Schieweck †

Abstract

We propose a Scott-Zhang type interpolation operator for the approximation of non-smooth functions in $H^1$ by means of continuous piecewise polynomials of low order. The novelty of the proposed interpolation operator is that it is defined for a general non-affine family of quadrilateral and hexahedral meshes with possibly hanging nodes. The derived interpolator exhibits optimal approximation properties for functions in $H^1$ and preserves homogeneous Dirichlet boundary conditions naturally.

Keywords: finite-element interpolation, hanging nodes, non-smooth functions

Mathematics Subject Classification (2000): 65N15, 65N30, 65N50

1 Introduction

The approximation properties of finite elements in Sobolev spaces have been well studied in general in e.g. [6, 4, 3] and references therein as well as in the case of general quadrilateral finite elements in [2]. In most cases, the proof of these approximation properties is based on the construction of an interpolation operator which uses nodal values of the function to be approximated. However, the nodal value of a function may not be well defined if the

*Institut für Angewandte Mathematik, Universität Heidelberg, INF 293, D-69120 Heidelberg, Germany, URL: http://gaia.iwr.uni-heidelberg.de/~heuvelin/
†Institut für Analysis und Numerik, Otto-von-Guericke-Universität Magdeburg, Postfach 4120, D-39016 Magdeburg, Germany, URL: http://www-ian.math.uni-magdeburg.de/home/schieweck/
function under consideration is too "rough". For example, functions in the Sobolev space $H^1$ have no pointwise value in two or more dimensions.

In [7, 17] an interpolation operator is constructed using patchwise averaging to define nodal values for functions in $L^1$. However, with this approach it is problematic to realize non-homogeneous boundary data. In [15], an interpolation operator for $H^1$ functions is constructed based on averaging over $(d-1)$-dimensional faces of the $d$-dimensional elements. The advantage of this approach in contrast to that of [7, 17] is that it preserves piecewise polynomial boundary conditions of the approximated function in a natural way.

The restriction of both approaches in [7, 17] as well as in [15] is that they assume a regular mesh and affine equivalent elements. The assumption of a regular mesh excludes the usage of meshes with so-called hanging nodes. From the practical point of view, meshes with hanging nodes are very attractive in applications with adaptive mesh refinement, in particular in the 3D-case (see e.g. [9, 10]). In [14], one can find an analysis for mixed hp-FEM with hanging nodes. The authors derive a Clément type interpolant for a special two-dimensional case with so-called geometric meshes where the elements are assumed to be affine equivalent quadrilaterals. For quadrilateral or hexahedral meshes, the assumption of affine equivalent elements means that only such elements are admitted that are parallelograms or parallelepipeds, respectively. Thus, from the practical point of view, this assumption is too restrictive.

In this note, we construct and analyze an interpolation operator for "rough", non-smooth functions in $H^1$. Such interpolation operators play an important role for the development of a posteriori error estimators (see e.g. [16, 17, 1] and references therein) as well as for the proof of the inf-sup condition in the context of mixed finite elements [8, 5]. This last application in the context of hp-FEM on general meshes with hanging nodes has been the starting point of this work [11]. Despite its high relevance in practice, little attention has been paid to the derivation of such interpolation operators for the case of general quadrilateral and hexahedral meshes with non-affine equivalent elements and with the freedom of having hanging nodes. Our goal in this paper is to fill this gap. We derive an interpolation operator with the following features:

- optimal order interpolation for functions in $H^1$;

- two and three dimensional meshes consisting of elements mapped by means of a multi-linear reference transformation, i.e. non-affine equivalent elements;
- possibility to use hanging nodes under the assumption of a so-called 1-regular mesh;
- preserving of homogeneous boundary conditions in a natural way.

For the construction and analysis of our operator, we use the general concept of defining appropriate nodal functionals (see e.g. [13]). Note that in the three-dimensional case of non-affine equivalent elements, the two dimensional faces of the hexahedral elements can be curved in general. This fact causes some extra difficulties in the analysis of the interpolation operator.

The outline of this paper is as follows: In §2, we introduce the notations and state the assumptions to be made on the meshes and finite element spaces. In §3 we prove some specific properties of general quadrilateral and hexahedral meshes with hanging nodes. These properties are needed for the analysis of the proposed interpolation operator. The construction and the proof of optimal local estimates of the interpolation error in $H^1$ and $L^2$ are established in §4.

# 2 Preliminaries and Notation

## 2.1 General Notations

We consider simultaneously the case of scalar and vector-valued functions $v : \Omega \rightarrow \mathbb{R}^n$ where $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is a bounded domain and $n = 1$ for scalar functions and $n = d$ for vector-valued functions. For a measurable set $G \subset \Omega$, let $(\cdot, \cdot)_G$ and $\| \cdot \|_{0,G}$ denote the inner product and the norm in $L^2(G)$ or $(L^2(G))^n$, respectively. Furthermore, let $| \cdot |_{m,G}$ and $\| \cdot \|_{m,G}$ denote the seminorm and norm in the Sobolev space $H^m(G)$ and $(H^m(G))^n$, respectively. For a $(d - 1)$-dimensional subset $E \subset \partial G$, we denote by $\langle \cdot, \cdot \rangle_E$ the inner product in $L^2(E)$, i.e.

$$\langle u, v \rangle_E := \int_E u(x)v(x) \, ds.$$  

We denote by $P_m(G)$ the space of all polynomials on the domain $G \subset \mathbb{R}^d$ with total degree less or equal to $m$ and by $Q_m(G)$ the space of those polynomials where the maximum power in each coordinate is less or equal to $m$.

By $\text{card}(J)$ we denote the number of elements of a finite set $J$. The Euclidean norm of a vector $v$ will be denoted by $\|v\|$. For a set $G \subset \mathbb{R}^d$, we denote by $\text{int}(G)$ and $\overline{G}$ the interior and closure of $G$, respectively. If $G \subset \Omega$ is a $d$-dimensional set then we will write
\(|G|\) for the \(d\)-dimensional measure of \(G\) and if \(E \subset \partial G\) is a \((d - 1)\)-dimensional subset we denote by \(|E|\) the \((d - 1)\)-dimensional measure of \(E\). The meaning will be clear from the context. Throughout this paper, \(C, C', \bar{C}\) will denote generic constants which may have different values at different places. All these constants occurring inside of any estimates will be independent of the local and global mesh parameter \(h_K\) and \(h\), respectively, which will be defined below.

### 2.2 Meshes and finite-element spaces

Let the bounded domain \(\Omega \subset \mathbb{R}^d\) be decomposed by a mesh \(T\) of elements \(K \in T\) which are assumed to be open quadrilaterals in the 2D-case and open hexahedrons in the 3D-case such that \(\Omega = \bigcup_{K \in T} K\). For an element \(K \in T\) we denote by \(h_K\) the diameter of the element \(K\) and by \(\rho_K\) the diameter of the largest ball that can be inscribed into \(K\). The meshwidth \(h\) of \(T\) is given by \(h := \max_{K \in T} h_K\). We assume that the mesh is shape regular in the sense of \([12]\). Note that for general non-affine families of quadrilateral or hexahedral meshes the usual shape regularity assumption \(\frac{h_K}{\rho_K} \leq C\) for all \(K \in T\) is not sufficient. The shape regularity assumption of \([12]\) imposes that the distortion of the quadrilateral or hexahedral elements from a parallelogram or parallelepiped, respectively, is uniformly bounded (see below for more details). This guarantees that the mapping \(F_K : \hat{K} \to K\) between the reference element \(\hat{K} = (-1,1)^d\) and the original element \(K \in T\) is bijective.

Note that the mapping \(F_K\) is multi-linear, i.e. \(F_K \in (Q_1(\hat{K}))^d\), which implies that in the three dimensional case \(d = 3\) the faces of a hexahedral element are in general curved. The corresponding normal unit vectors are therefore not constant.

In the following, we describe the shape regularity assumption of \([12]\) in more detail. By a Taylor expansion of \(F_K(\hat{x})\) we get

\[
F_K(\hat{x}) = b_K + B_K \hat{x} + G_K(\hat{x}),
\]

with \(b_K := F_K(0)\), \(B_K := DF_K(0)\) and \(G_K(\hat{x}) := F_K(\hat{x}) - F_K(0) - DF_K(0)(\hat{x})\). We denote by \(\hat{S} \subset \hat{K}\) the \(d\)-simplex with the vertices \((0,\ldots,0)\), \((1,0,\ldots,0)\), \ldots, \((0,\ldots,0,1)\) and by \(S_K\) the image of \(\hat{S}\) under the affine mapping \(\hat{x} \to B_K \hat{x} + b_K\). For the simplices \(S_K, K \in T\), we assume the usual shape regularity assumption

\[
\frac{h_{S_K}}{\rho_{S_K}} \leq C \quad \forall K \in T,
\]
Interpolation operator for \(H^1\) functions

where \(h_{S_K} := \text{diam}(S_K)\) and \(\rho_{S_K}\) is the diameter of the largest ball inscribed into \(S_K\). Note that we have

\[
\|B_K\| \leq Ch_{S_K}, \quad \|B_K^{-1}\| \leq Ch_{S_K}^{-1} \quad \forall K \in T,
\]

where \(\|B_K\|\) denotes the matrix norm induced by the Euclidean vector norm in \(\mathbb{R}^d\). For each element \(K \in T\), we define the constant \(\gamma_K\) by

\[
\gamma_K := \sup_{\hat{x} \in \hat{K}} \|B_K^{-1} DF_K(0) - I\|,
\]

which is a measure of the deviation of \(K\) from a parallelogram or a parallelepiped, respectively. Note that \(\gamma_K = 0\) if the mapping \(F_K\) is affine.

Definition 1 A mesh \(T\) of quadrilateral or hexahedral elements is called shape regular if the conditions (2) and

\[
\gamma_K \leq \gamma_0 < 1 \quad \forall K \in T,
\]

are satisfied.

In [12], it has been proven that for a shape regular mesh in the sense of Definition 1, the following estimates hold

\[
d! (1 - \gamma_K)^d |S_K| \leq |\det(DF_K(\hat{x}))| \leq d! (1 + \gamma_K)^d |S_K| \quad \forall \hat{x} \in \hat{K},
\]

\[
|DF_K(\hat{x})| \leq (1 + \gamma_K) \|B_K\| \quad \forall \hat{x} \in \hat{K}.
\]

Furthermore, in [12] one can find sufficient conditions that guarantee the assumption (5). These conditions can be easily checked in practical computations. In [2] some conditions for general two-dimensional quadrilateral meshes are given in order to derive optimal approximation estimates of the corresponding \(Q_r\) finite element spaces.

In this paper, our main interest is to derive an \(H^1\)-stable first order interpolation operator for meshes with hanging nodes. That means that the usual assumption of a regular grid \(T\) has to be weakened.

In the following, we will describe the type of grids that is treated in this paper. \(T\) is a multi-level grid generated by a refinement process in the following way. We start with a partition \(T^0\) of the domain \(\Omega\) into elements \(K \in T^0\) of grid-level 0, i.e. \(\Omega = \text{int}(\bigcup_{K \in T^0} K)\). The grid \(T^0\) is assumed to be regular in the usual sense, i.e. for any two different elements \(K_1, K_2 \in T^0\) the intersection \(\overline{K_1} \cap \overline{K_2}\) is either empty or a common \((d - m)\)-dimensional face of \(K_1\) and \(K_2\) where \(m \in \{1, \ldots, d\}\). Now, starting with the elements \(K \in T^0\),
Figure 1: refinement of element $K$ into son-elements $\sigma_i(K)$, $i = 1, \ldots, 4$

an existing element $K$ can be refined, i.e. it can be splitted into $2^d$ many new elements called son-elements and denoted by $\sigma_i(K)$, $i = 1, \ldots, 2^d$ (see Figure 1). For a new element $K' = \sigma_i(K)$, we will say that $K$ is the father-element of $K'$ and we will write $K = \mathcal{F}(K')$. If an element $K$ is refined then, in the partition of the domain $\Omega$, it is replaced by the set of its son-elements $\sigma_i(K)$, $i = 1, \ldots, 2^d$. The new elements can be refined again and again and so the final partition $T$ of $\Omega$ is created. Examples of such grids used in practical computations can be found in [9], [10].

**Definition 2** For an element $K \in T$, generated from the initial grid $T^0$ by the refinement process described above we define the refinement level $\ell(K)$ as $\ell(K) := 0$ if $K \in T^0$ and $\ell(K) := m \geq 1$ if there exists a chain of $m$ father-elements $K_i$, $i = 1, \ldots, m$, starting from $K_0 := K$ and defined by $K_i := \mathcal{F}(K_{i-1})$ for $i = 1, \ldots, m$, such that $K_m \in T^0$.

The above defined refinement level $\ell(K)$ is equal to the number of refinement steps that is needed to generate element $K$ from an element of the coarsest grid $T^0$.

**Definition 3** A grid $T$, generated by the above defined refinement process from the initial grid $T^0$, is called 1-regular if it holds

$$|\ell(K) - \ell(K')| \leq 1$$

for any pair of face-neighbored elements $K, K' \in T$ where the $(d-1)$-dimensional measure of $\partial K \cap \partial K'$ is positive.

In this paper, we consider only grids $T$ which are 1-regular. For practical computations, this is not a restriction since complicated structures can be described in a reasonable way by means of 1-regular grids (see e.g. [9], [10]).

Finally, we need some additional notation for the analysis in the following sections. We denote by $\mathcal{E}(K)$ the set of all $(d-1)$-dimensional faces of an element $K$, by $n^K$ the unit
Interpolation operator for $H^1$ functions

Figure 2: Two dimensional configuration for the case of a regular inner face i.e. $E \in \mathcal{E}_r$ (left) and for the case of an irregular inner face i.e. $E \in \mathcal{E}_i$ (right) where $E_1, E_2$ are the son-faces of $E$.

normal vector on the element boundary $\partial K$ directed outward with respect $K$ and by $n^K_E$ the restriction of the normal vector $n^K$ to the face $E \in \mathcal{E}(K)$. Let $\mathcal{E}$ be the set of all faces $E \in \mathcal{E}(K)$ of all elements $K \in T$. We split $\mathcal{E}$ in the form

$$\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}(\Gamma)$$

where $\mathcal{E}(\Gamma)$ describes all faces of $\mathcal{E}$ located at the boundary $\Gamma$ of $\Omega$ and $\mathcal{E}_0$ denotes the set of inner faces of $\mathcal{E}$. For any face $E \in \mathcal{E}$, we define the set $T(E)$ of cells associated with $E$ as

$$T(E) := \{K \in T : E \in \mathcal{E}(K)\}.$$

Let $\mathcal{E}_r$ and $\tilde{\mathcal{E}}_r$ denote the set of the regular inner faces and the set of the regular faces defined as

$$\mathcal{E}_r := \{E \in \mathcal{E}_0 : \text{card}(T(E)) = 2\} \quad \text{and} \quad \tilde{\mathcal{E}}_r := \mathcal{E}_r \cup \mathcal{E}(\Gamma),$$

respectively. For each regular face $E \in \mathcal{E}_r$, there exist exactly two different elements denoted by $K(E)$ and $K'(E)$ such that $E$ is one of their faces, i.e.

$$T(E) = \{K(E), K'(E)\} \quad \forall E \in \mathcal{E}_r.$$

For all other faces $E \in \mathcal{E} \setminus \mathcal{E}_r$, there is only one element denoted by $K(E)$ which has $E$ as one of its faces, i.e.

$$T(E) = \{K(E)\} \quad \forall E \in \mathcal{E} \setminus \mathcal{E}_r.$$

A face $\tilde{E} \in \mathcal{E}$ is called a son-face of a face $E \in \mathcal{E}$ if $\tilde{E} \subset E$ and $|\tilde{E}| < |E|$ where $|\tilde{E}|$ and $|E|$ denote the $(d - 1)$-dimensional measure of $\tilde{E}$ and $E$, respectively (see Figure 2 We denote
by $\sigma(E)$ the set of all son-faces of $E$. Note that for each regular face $E \in \mathcal{E}_r$, the set $\sigma(E)$ is empty. We denote by $\mathcal{E}_i$ the set of all irregular inner faces defined as

$$\mathcal{E}_i := \{ E \in \mathcal{E}_0 : \sigma(E) \neq \emptyset \}.$$ 

Using these definitions, the set $\mathcal{E}_0$ of all inner faces can be decomposed as

$$\mathcal{E}_0 = \mathcal{E}_r \cup \mathcal{E}_i \cup \left( \bigcup_{E \in \mathcal{E}_i} \sigma(E) \right).$$

Let $\tilde{E} \in \sigma(E)$ be a son-face of $E \in \mathcal{E}_i$, then the face $E$ is called the father-face of $\tilde{E}$ and we will write $E = \mathcal{F}(\tilde{E})$. We define the set of all son-faces by

$$\mathcal{E}_\sigma := \bigcup_{E \in \mathcal{E}_i} \sigma(E).$$

For a given element $K \in \mathcal{T}$, we define further

$$\mathcal{E}_\mu(K) := \mathcal{E}(K) \cap \mathcal{E}_\mu \quad \text{for } \mu \in \{ \text{r, i, } \sigma \}.$$ 

Finally, for each face $E \in \mathcal{E}$, let $n_E$ denote the unit vector $n_E^{K(E)}$ which is normal to the face $E$ and directed outward with respect to the element $K(E)$.

For the subsequent analysis, we need the scalar finite element spaces $\tilde{S}_h^1 \subset H^1(\Omega)$ and $S_h^1 \subset H_0^1(\Omega)$ associated with the mesh $\mathcal{T}$

$$\tilde{S}_h^1 := \{ \phi \in H^1(\Omega) : \phi|_K \circ F_K \in Q_1(\hat{K}) \quad \forall K \in \mathcal{T} \}, \quad (8)$$

$$S_h^1 := \tilde{S}_h^1 \cap H_0^1(\Omega), \quad (9)$$

and the finite element spaces of vector-valued functions $v : \Omega \rightarrow \mathbb{R}^n$ defined as $\tilde{X}_h^1 := (\tilde{S}_h^1)^n$ and $X_h^1 := (S_h^1)^n$. Some attention is required to ensure interelement continuity in (8) and (9) in case of hanging nodes. We refer to section 4.1 for the treatment of this issue.

3 Some properties of the meshes

**Lemma 4** Let $\mathcal{T}$ be a 1-regular mesh which is shape regular in the sense of Definition 1. Furthermore, let $h_{S_K}$ denote the diameter of simplex $S_K$ associated to the element $K \in \mathcal{T}$ as described in Section 2.2. Then, for each element $K \in \mathcal{T}$, the following estimates are satisfied

$$Ch_K \leq h_{S_K} \leq C'h_K, \quad (10)$$
\[ \|DF_K(\hat{x})\| \leq Ch_K \quad \forall \hat{x} \in \hat{K}, \]  
(11)

\[ Ch_K^d \leq |\det(DF_K(\hat{x}))| \leq C'h_K^d \quad \forall \hat{x} \in \hat{K}. \]  
(12)

**Proof.** In order to prove (10), we consider the affine mapping

\[ F^\text{aff}_K(\hat{x}) := F_K(0) + DF_K(0)\hat{x} = b_K + B_K\hat{x}, \]

which maps \( \hat{S} \) to \( S_K \). Then, we get

\[ h_{S_K} = \sup_{\hat{x}, \hat{y} \in \hat{S}} \|F^\text{aff}_K(\hat{x}) - F^\text{aff}_K(\hat{y})\| \leq 2\sqrt{d} \|B_K\|. \]  
(13)

A direct computation shows that the column vectors of the matrix \( B_K \) can be represented as a linear combination of differences of the vectors \( a_{K,m} = F_K(\hat{a}_m), \ m = 1, \ldots, 2^d \), where \( \hat{a}_m \) are the vertices of \( \hat{K} = (-1,1)^d \). The vectors \( a_{K,m} \) correspond to the vertices of the original element \( K \). Therefore, we obtain the estimate \( \|B_K\| \leq Ch_K \) which implies by means of (13) the estimate \( h_{S_K} \leq C'h_K \). Using the estimates (7), (5) and (3) we get

\[ h_K = \sup_{\hat{x}, \hat{y} \in \hat{K}} |F_K(\hat{x}) - F_K(\hat{y})| \leq 2\sqrt{d} \sup_{\xi \in \hat{K}} \|DF_K(\xi)\| \]
\[ \leq C(1 + \gamma_K)\|B_K\| \leq C(1 + \gamma_0)h_{\hat{S}_K}, \]

which proves the estimate \( Ch_K \leq h_{S_K} \).

Now, the estimate (11) is a simple consequence of (7), (5), (3) and the estimate \( h_{S_K} \leq C'h_K \). Using the estimates (6), (5), (2) and (10) we get

\[ |\det(DF_K(\hat{x}))| \geq d!(1 - \gamma_0)^d |S_K| \geq C\rho_{\hat{S}_K}^d \geq Ch_{\hat{S}_K}^d \geq Ch_K^d \quad \forall \hat{x} \in \hat{K}, \]

which proves the lower bound in (12). The upper bound in (12) is a simple consequence of (11). \( \square \)

**Lemma 5** Let \( T \) be a 1-regular mesh which is shape regular in the sense of Definition 1. Then for any element \( K \in T \) and for any face \( E \in \mathcal{E}(K) \) the following estimates hold

\[ Ch_{K}^{d-1} \leq |E| \leq C'h_{K}^{d-1}. \]  
(14)
**Proof.** Let \( \hat{a}_m, m = 1, \ldots, 2^d \) describe the vertices of the reference element \( \hat{K} := (-1,1)^d \) in \( \mathbb{R}^d \).

First we consider the two dimensional case \( d = 2 \). Clearly there exist \( i_1, i_2 \in \{1, 2, 3, 4\} \) such that

\[
|E| = |F_K(\hat{a}_{i_1}) - F_K(\hat{a}_{i_2})| = |DF_K(\xi) (\hat{a}_{i_1} - \hat{a}_{i_2})| = 2|DF_K(\xi) e^m|,
\]

where \( \xi \in \hat{K} \) and \( e^m \) for \( m = 1, 2 \) denotes the \( m \)-th unit vector in \( \mathbb{R}^2 \). Let \( d_1, d_2 \in \mathbb{R}^2 \) be defined such that \( DF_K(\xi) = [d_1, d_2] \). It is well known that \( |\det([d_1, d_2])| \) corresponds to the area of the parallelogram spanned by the vectors \( d_1 \) and \( d_2 \), i.e.

\[
|\det(DF_K(\xi))| = \|d_1\| \|d_2\| \sin(\alpha),
\]

where \( \alpha \) is the angle between \( d_1 \) and \( d_2 \). Then, based on the estimate (12) of Lemma 4 we obtain

\[
\|d_1\| \geq \frac{Ch_K^2}{\|d_2\| \sin(\alpha)} \geq Ch_K.
\]

Similarly we derive \( \|d_2\| \geq Ch_K \) and combined with (15) this leads to \( |E| \geq Ch_K \).

For the three dimensional case \( d = 3 \) the proof is slightly more involved. We assume \( E \) to be defined such that \( \hat{E} = F_K^{-1}(E) \) corresponds to the face \( \hat{x}_1 = -1 \). Then, we can represent each point \( \hat{x} \in \hat{E} \) as

\[
\hat{x} = \hat{\gamma}(t_1, t_2) := (-1, t_1, t_2)^T \quad \text{where} \quad (t_1, t_2) \in G := (-1,1)^2,
\]

and each point \( x \in E \) as

\[
x = \gamma(t_1, t_2) := F_K(\hat{\gamma}(t_1, t_2)), \quad \forall (t_1, t_2) \in G.
\]

Further, we have

\[
|E| = \int_G \|N_E(t_1, t_2)\| \, dt_1 \, dt_2, \tag{16}
\]

with

\[
N_E(t_1, t_2) := \frac{\partial \gamma}{\partial t_1} \times \frac{\partial \gamma}{\partial t_2} \big|_{(t_1, t_2)} = \frac{\partial F_K}{\partial \hat{x}_2} \times \frac{\partial F_K}{\partial \hat{x}_3} \big|_{\hat{x} = \hat{\gamma}(t_1, t_2)}.
\]

By definition we have

\[
det(DF_K(\hat{\gamma}(t_1, t_2))) = \frac{\partial F_K}{\partial \hat{x}_1} \cdot \left( \frac{\partial F_K}{\partial \hat{x}_2} \times \frac{\partial F_K}{\partial \hat{x}_3} \right) \big|_{\hat{x} = \hat{\gamma}(t_1, t_2)}
\]

\[
= \frac{\partial F_K}{\partial \hat{x}_1} \cdot N_E(t_1, t_2) = \|\frac{\partial F_K}{\partial \hat{x}_1}\| \cdot \|N_E(t_1, t_2)\| \cdot \cos(\alpha), \tag{17}
\]

Where \( \alpha \) is the angle between \( \frac{\partial F_K}{\partial \hat{x}_1} \) and \( N_E(t_1, t_2) \).
where $\alpha$ is the angle between $\frac{\partial F}{\partial \hat{x}_1}$ and $N_E(t_1, t_2)$. From (17) we deduce

$$
\|N_E(t_1, t_2)\| = \frac{\text{det}(DF_K(\hat{\gamma}(t_1, t_2)))}{\|\frac{\partial F}{\partial \hat{x}_1}\| \cdot \cos(\alpha)}
$$

(18)

Then, from (12) and (11) we get

$$
\|N_E(t_1, t_2)\| \geq Ch^2_K.
$$

(19)

Together with (16) we get

$$
|E| \geq Ch^2_K,
$$

which was to be proved. Obviously the proof where $\hat{\gamma}$ corresponds to $\hat{x}_i = \pm 1$ is completely analogous. □

Lemma 6 Let $E \in \mathcal{E}_r \cup \mathcal{E}_i$ and $K \in \mathcal{T}$ such that $E \in \mathcal{E}(K)$. Then, the following estimate holds

$$
\|v\|_{L^2(E)} \leq C \ h^{1/2}_K \left\{ h^{-1}_K \|v\|_{0,K} + |v|_{1,K} \right\} \quad \forall v \in (H^1(K))^d.
$$

(20)

Proof. At first, we need to prove the estimate

$$
\|v\|_{L^2(E)} \leq C \ h^{(d-1)/2}_K \|\hat{v}\|_{L^2(\tilde{E})},
$$

(21)

where the function $\hat{v} \in (H^1(\tilde{K}))^d$ is defined by $\hat{v} := v(F_K(\hat{x}))$ for all $\hat{x} \in \hat{K}$. We present the proof of (21) only for the three dimensional case $d = 3$. The case $d = 2$ follows easily by simple transformation of the integrals corresponding to both sides of (21).

We consider only the special case where $\tilde{E} = F_1^{-1}(E)$ corresponds to $\hat{x}_1 = -1$. For the face $E$, we get

$$
\|v\|^2_{L^2(E)} = \int_G v(\gamma(t_1, t_2)) \|N_E(t_1, t_2)\| \, dt_1 \, dt_2,
$$

with

$$
N_E(t_1, t_2) = \left(\frac{\partial \gamma}{\partial t_1} \times \frac{\partial \gamma}{\partial t_2}\right)_{(t_1, t_2)}.
$$

Now, considering the estimate (11),

$$
\|N_E(t_1, t_2)\| \leq \left\|\frac{\partial F_K}{\partial \hat{\gamma}_2}(\hat{\gamma}(t_1, t_2))\right\| \cdot \left\|\frac{\partial F_K}{\partial \hat{\gamma}_3}(\hat{\gamma}(t_1, t_2))\right\| \leq Ch^2_K,
$$

we obtain

$$
\|v\|^2_{L^2(E)} \leq Ch^2_K \int_G v(\gamma(t_1, t_2))^2 \, dt_1 \, dt_2.
$$

(22)
The parametrization of \( \hat{E} \) is given by \( \hat{x} = \hat{\gamma}(t_1, t_2) = (-1, t_1, t_2)^T \) for \((t_1, t_2) \in G\) and the corresponding normal vector is

\[
\hat{N}_E(t_1, t_2) := \left( \frac{\partial \hat{\gamma}}{\partial t_1} \times \frac{\partial \hat{\gamma}}{\partial t_2} \right) \bigg|_{(t_1, t_2)} = (1, 0, 0)^T.
\]

Therefore, we get

\[
\|\hat{v}\|_{L^2(\hat{E})}^2 = \int_G \hat{v}(\hat{\gamma}(t_1, t_2))^2 \|\hat{N}_E(t_1, t_2)\| dt_1 dt_2 = \int_G v(\gamma(t_1, t_2))^2 dt_1 dt_2.
\]

Together with (22), this proves (21) for the case \( d = 3 \). Now, we apply the trace theorem on the reference element \( \hat{K} \) and well-known estimates between the norms of \( \hat{v} \) and \( \hat{K} \) and the norms of \( v \) on \( K \) (see e.g. [6]) and get

\[
\|\hat{v}\|_{L^2(\hat{E})} \leq C\|\hat{v}\|_{0,\hat{K}} + C|\hat{v}|_{1,\hat{K}} \leq C h_K^{-d/2} \|v\|_{0,K} + C h_K^{1-d/2} |v|_{1,K}.
\]

The proof where \( \hat{E} \) corresponds to \( \hat{x}_i = \pm 1 \) is completely analogous. Together with (21), this proves the estimate (20). \( \square \)

## 4 Interpolation operator

### 4.1 Construction of the operator \( R_h \)

Let \( \hat{a}_m, m = 1, \ldots, 2^d \), denote the vertices of the reference element \( \hat{K} := (-1, +1)^d \). We introduce the global index set \( J^1 \) of all vertices \( \{a_j\}_{j \in J^1} \) of the mesh \( T \). An index \( j \in J^1 \) will be called a node and the associated vertex \( a_j \) a nodal point. Obviously, for each nodal point \( a_j \), there exists at least one element \( K \in T \) such that \( a_j = F_K(\hat{a}_m) \) where \( m \in \{1, \ldots, 2^d\} \).

Let us define by

\[
J^1(K) := \{ j \in J^1 : a_j = F_K(\hat{a}_m), \ 1 \leq m \leq 2^d \}.
\]

the set of nodes associated with the element \( K \in T \). An index \( j \in J^1 \) is called a hanging node if there exists an irregular face \( E \in \mathcal{E}_i \) and a son face \( \tilde{E} \in \sigma(E) \) such that

\[
j \notin J^1(K(E)) \quad \text{and} \quad j \in J^1(K(\tilde{E})).
\]

The set of indices associated to regular (i.e. non-hanging) nodes is denoted by \( J^1_r \).
Interpolation operator for $H^1$ functions

Following the idea in Scott-Zhang [15], we assign to each nodal point $a_j$, $j \in \{1, \ldots, n\}$, a face $E(j) := E_j \in \tilde{\mathcal{E}}_r \cup \mathcal{E}_i$, such that

$$a_j \in E_j \cup \partial E_j,$$  \hspace{1cm} (23)

$$a_j \in \partial \Omega \Rightarrow E_j \subset \partial \Omega,$$  \hspace{1cm} (24)

$$j \in J^1 \setminus J^1_r \Rightarrow E_j \in \mathcal{E}_i.$$  \hspace{1cm} (25)

Note that the conditions (23-25) do not lead to a unique possible mapping $E(\cdot)$.

For $j \in J^1_r$, let $\varphi_j \in \tilde{S}_h^1$ denote the basis function associated with the regular vertex node $a_j$ by the condition

$$\varphi_j(a_i) = \delta_{ij} \quad \forall i \in J^1_r.$$  \hspace{1cm} (26)

Then, due to continuity requirements, the nodal values $\varphi(a_j)$ of a finite element function $\varphi \in \tilde{S}_h^1$ at a hanging node $a_j$ is determined by a linear combination of the nodal values $\varphi(a_i)$ for $i \in J^1(K(E_j))$ i.e.

$$\varphi(a_j) = \sum_{i \in J^1(K(E_j))} \alpha_{j,i} \varphi(a_i) \quad \forall \varphi \in \tilde{S}_h^1,$$  \hspace{1cm} (27)

where

$$\alpha_{j,i} := \varphi_i(a_j) \quad \forall j \in J^1 \setminus J^1_r, \quad i \in J^1(K(E_j)).$$  \hspace{1cm} (28)

Now, for the definition and also for the subsequent analysis of the interpolation operator, we use the general concept of defining suitable nodal functionals [13]. For $j \in J^1_r$ and $v := (v_1, \ldots, v_n)^T \in (H^1(\Omega))^n$, we define the vector-valued nodal functional $N_j(v) := (N_{j,1}(v), \ldots, N_{j,n}(v))^T \in \mathbb{R}^n$ with

$$N_{j,k}(v) := |E_j|^{-1} \int_{E_j} v_k \, ds \quad \forall j \in J^1_r, \quad k \in \{1, \ldots, n\}.$$  \hspace{1cm} (29)

Based on this definition and on the definition (28) we define further for $j \in J^1 \setminus J^1_r$ the vector-valued functional $\tilde{N}_{j,k} : (H^1(\Omega))^n \rightarrow \mathbb{R}$ by means of

$$\tilde{N}_{j,k}(v) := \sum_{i \in J^1(K(E_j))} \alpha_{j,i} N_{i,k}(v) \quad \forall j \in J^1 \setminus J^1_r, \quad k \in \{1, \ldots, n\}.$$  \hspace{1cm} (30)

For a given function $v \in (H^1(\Omega))^n$, we define the interpolate $R_h v \in \bar{X}_h^1$ locally on each element $K \in T$ as

$$R_h v|_K := \sum_{j \in J^1(K)} N_j(v) \psi_j^K + \sum_{j \in J^1(K) \setminus J^1_r(K)} \tilde{N}_j(v) \psi_j^K,$$  \hspace{1cm} (31)
where $\psi^K_j$ is the usual $Q_1$ Lagrange basis function on $K$ with respect to the node $a_j$ for $j \in J^1(K)$. One can easily show that this elementwise definition yields a globally continuous function, i.e. for $v \in (H^1(\Omega))^n$ we have $R_h v \in \tilde{X}_h^1$. Due to the condition (25) we have furthermore

$$R_h v \in X_h^1 \subset (H^1_0(\Omega))^n \quad \forall v \in (H^1_0(\Omega))^n.$$  \hfill (32)

In the same way, one can easily show the property

$$R_h v = v \quad \text{if} \quad v(x) = c \in \mathbb{R}^n \quad \forall x \in \Omega.$$  \hfill (33)

### 4.2 Properties of the operator $R_h$

For an element $K \in T$, we define the set of neighboring elements of $K$ as

$$\Lambda(K) := \left\{ \tilde{K} \in T : K \cap \tilde{K} \neq \emptyset \right\}.$$  \hfill (34)

**Lemma 7** Let $T$ be a 1-regular and shape regular mesh. Then for all $K \in T$, it holds

$$Ch_{\tilde{K}} \leq h_K \leq C' h_{\tilde{K}} \quad \forall \tilde{K} \in \Lambda(K).$$  \hfill (35)

**Proof.** It is sufficient to show the estimate $h_K \leq C' h_{\tilde{K}}$ since the estimate $Ch_{\tilde{K}} \leq h_K$ follows by exchanging the role of $K$ and $\tilde{K}$. For each $\tilde{K} \in \Lambda(K)$, there exists a chain of face neighbored elements $K_i$, $i = 0, \ldots, s$, with $K_0 = K$ and $K_s = \tilde{K}$ such that for the common boundary $E_i := \partial K_{i-1} \cap \partial K_i$ one the following three conditions is satisfied for $i = 1, \ldots, s$:

- **Case 1:** $E_i \in \mathcal{E}(K_{i-1})$ and $E_i \in \mathcal{E}(K_i)$,
- **Case 2:** $E_i \in \mathcal{E}_\sigma(K_{i-1})$ and $\mathcal{F}(E_i) \in \mathcal{E}(K_i)$,
- **Case 3:** $\mathcal{F}(E_i) \in \mathcal{E}(K_{i-1})$ and $E_i \in \mathcal{E}(K_i)$,

where $\mathcal{F}(E_i)$ denotes the father face of $E_i$. Using (14), we get for each $i \in \{1, \ldots, s\}$ the estimates

- **Case 1:** $h_{K_{i-1}} \leq C |E_i|^{1/(d-1)} \leq \tilde{C} h_{K_i}$,
- **Case 2:** $h_{K_{i-1}} \leq C |E_i|^{1/(d-1)} \leq C |\mathcal{F}(E_i)|^{1/(d-1)} \leq \tilde{C} h_{K_i}$,
- **Case 3:** $h_{K_{i-1}} \leq C |\mathcal{F}(E_i)|^{1/(d-1)} \leq C |E_i|^{1/(d-1)} \leq \tilde{C} h_{K_i}$.
Thus, we obtain \( h_K \leq \tilde{C}^s h_{\tilde{K}} \). Due to the shape regularity of the mesh \( T \), the number \( s \) of the chain connecting \( K \) with \( \tilde{K} \in \Lambda(K) \) is uniformly bounded which implies the estimate \( h_K \leq C' h_{\tilde{K}} \). □

We are now able to state the main result of this paper.

**Theorem 8** For \( K \in T \), the operator \( R_h \) defined in (31) satisfies

\[
|R_h v|_{1,K} \leq C|v|_{1,\omega(K)} \quad \forall v \in (H^1(\Omega))^n, \tag{36}
\]

\[
\|v - R_h v\|_{0,K} \leq Ch_K|v|_{1,\omega(K)} \quad \forall v \in (H^1(\Omega))^n, \tag{37}
\]

where

\[
\delta(K) := \bigcup_{\tilde{K} \in \Lambda(K)} \tilde{K}, \quad \omega(K) := \bigcup_{\tilde{K} \in \Lambda(K)} \delta(\tilde{K}).
\]

**Proof.** First we derive some estimates of \( \|N_j\| \) for \( j \in J_1^t(K) \) and \( \|\tilde{N}_j\| \) for \( j \in J_1^t(K) \setminus J_1^t(K) \) with respect to \( h_K \). Based on the definition (29), we obtain for \( j \in J_1^t \) and \( v \in (H^1(\Omega))^n \)

\[
\|N_j(v)\| \leq |E_j|^{-1} \int_{E_j} \|v\| \, ds \leq |E_j|^{-1/2} \|v\|_{L^2(E_j)},
\]

\[
\leq Ch_K^{1/2} \left\{ h_K^{-1} \|v\|_{0,\tilde{K}} + |v|_{1,\tilde{K}} \right\},
\]

\[
\leq Ch_K^{1/2} \left\{ h_K^{-1} \|v\|_{0,\delta(K)} + |v|_{1,\delta(K)} \right\}, \tag{38}
\]

where the last two estimates follow from (20) and (35) assuming \( \tilde{K} = K(E_j) \). Now based on the definition (30) of \( \tilde{N}_{j,k}(v) \) we obtain for \( j \in J_1^t(K) \) and for \( v \in (H^1(\Omega))^n \)

\[
\|\tilde{N}_j(v)\| \leq \sum_{i \in J_1^t(K(E_j))} |\alpha_{j,i}| \|N_i(v)\| \leq \sum_{i \in J_1^t(K(E_j))} \|N_i(v)\|, \tag{39}
\]

where the last inequality directly follows from the definition (28) of \( \alpha_{j,i} \) which obviously leads to \( |\alpha_{j,i}| \leq 1 \). Since \( K(E_j) \subset \delta(K) \), we deduce from (38) and (39)

\[
\|\tilde{N}_j(v)\| \leq Ch_K^{1-d} \left\{ h_K^{-1} \|v\|_{0,\omega(K)} + |v|_{1,\omega(K)} \right\} \quad \forall j \in J_1^t(K) \setminus J_1^t(K). \tag{40}
\]

From (31), (38) and (40) we obtain

\[
|R_h v|_{1,K} \leq C \left\{ \sum_{j \in J_1^t(K)} \|N_j(v)\| \left| \psi^j_{\omega(K)} \right|_{1,K} + \sum_{j \in J_1^t(K) \setminus J_1^t(K)} \|\tilde{N}_j(v)\| \left| \psi^j_{\omega(K)} \right|_{1,K} \right\}
\]

\[
\leq C \left\{ h_K^{-1} \|v\|_{0,\omega(K)} + |v|_{1,\omega(K)} \right\}, \tag{41}
\]

\[
\|v - R_h v\|_{0,K} \leq C h_K|v|_{1,\omega(K)} \quad \forall v \in (H^1(\Omega))^n.
\]
since
\[ |\psi^K_j|_{1,K} \leq Ch^{d-1}_K \quad \forall K \in T, \quad \forall j \in J^1(K). \]
For any constant \( c \in \mathbb{R}^n \) we obtain from (33) the equality
\[ |R_h v|_{1,K} = |R_h v - c|_{1,K} = |R_h (v - c)|_{1,K}, \]
which combined with (41) leads to
\[ |R_h v|_{1,K} \leq C \left\{ h^{-1}_K \| v - c \|_{0,\omega(K)} + |v|_{1,\omega(K)} \right\}. \]
Now let \( c \in \mathbb{R}^n \) be the constant \( L_2 \) interpolate of \( v \) in \( \omega(K) \), then we get the estimate (36) which is to be proved.

The interpolation property (37) is proven in a similar way. From (31), (38) and (40) we obtain
\[
|R_h v|_{0,K} \leq C \left\{ \sum_{j \in J^1(K)} \| \tilde{N}_j(v) \| \| \psi^K_j \|_{0,K} + \sum_{j \in J^1(K) \setminus J^1(K)} \| \tilde{N}_j(v) \| \| \psi^K_j \|_{0,K} \right\}
\leq C \left\{ \| v \|_{0,\omega(K)} + h_K |v|_{1,\omega(K)} \right\},
\]
where the last inequality follows from
\[ \| \psi^K_j \|_{0,K} \leq Ch^{d/2}_K \quad \forall K \in T, \quad \forall j \in J^1(K). \]
Let \( c \in \mathbb{R}^n \) be the constant interpolate of \( v \) on \( \omega(K) \) in the \( L_2 \) norm, then we obviously get
\[ \| R_h (v - c) \|_{0,K} \leq C \left\{ \| v - c \|_{0,\omega(K)} + h_K |v|_{1,\omega(K)} \right\} \leq Ch_K |v|_{1,\omega(K)}. \]
This estimate combined with (33) the directly leads to
\[ \| v - R_h v \|_{0,K} \leq \| v - c \|_{0,K} + Ch_K |v|_{1,\omega(K)} = Ch_K |v|_{1,\omega(K)} \quad \forall v \in (H^1(\Omega))^n. \]
This concludes the proof. \( \square \)

References

Interpolation operator for $H^1$ functions


