

A PRIORI ERROR ESTIMATES FOR THE FINITE ELEMENT DISCRETIZATION OF ELLIPTIC PARAMETER IDENTIFICATION PROBLEMS WITH POINTWISE MEASUREMENTS

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Abstract. We develop an a priori error analysis for the finite element Galerkin discretization of parameter identification problems. The state equation is given by an elliptic partial differential equation of second order with a finite number of unknown parameters, which are estimated using point-wise measurements of the state variable.

Key words. Parameter identification, finite elements, pointwise measurements, L^∞ -error estimates.

AMS subject classifications. 65K10, 65N30, 65N21, 49M25, 49K20.

1. Introduction. We consider parameter identification problems governed by an elliptic partial differential equation of second order. The finitely many unknown parameters are estimated using the measurements of point values of the *state variable*. Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain, $L^2(\Omega)$ the corresponding Lebesgue space with inner product and norm denoted by (\cdot, \cdot) and $\|\cdot\|_2$, respectively, and $H^m(\Omega)$ the Sobolev space of order $m \in \mathbb{N}$. With this notation, we set

$$V := \{v \in H^1(\Omega) \cap C(\overline{\Omega}) \mid v = 0 \text{ on } \partial\Omega\}.$$

The state variable $u \in V$ is determined by an elliptic partial differential equation (the *state equation*)

$$\begin{aligned} -\nabla \cdot (A(q)\nabla u) &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

for given Hölder continuous $f \in C^\alpha(\overline{\Omega})$, $\alpha \in (0, 1)$. Here, $q \in Q_0 \subset Q = \mathbb{R}^{n_p}$ denotes the set of unknown parameters and $A(q) = (A_{ij}(q))$ is a symmetric and positive definite 2×2 matrix with twice continuously differentiable entries $A_{ij} : Q \rightarrow C^{1+\alpha}(\overline{\Omega})$. The above conditions guarantee that for any admissible value of the parameter q , the corresponding solution u of the state equation (1.1) is in $H^2(\Omega)$. At the corner points of $\partial\Omega$, the second derivatives of the solution may become singular. However, u has Hölder continuous second derivatives, $u \in C^{2+\alpha}(\overline{\Omega}_d)$, for each subdomain $\Omega_d \subset \Omega$ with positive distance to the corner points.

The usual weak formulation of (1.1) is:

$$a(q)(u, \phi) = (f, \phi) \quad \forall \phi \in V, \tag{1.2}$$

where the bilinear form $a(q)(\cdot, \cdot)$ is defined by

$$a(q)(u, \phi) := (A(q)\nabla u, \nabla \phi). \tag{1.3}$$

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Further, the observation operator $C(\cdot)$ describing the mapping of the state variable u to the space of measurements $Z = \mathbb{R}^{n_m}$ is given by

$$C_i(v) = v(\xi_i), \quad i = 1, 2, \dots, n_m, \quad (1.4)$$

where $\{\xi_i\} \subset \Omega$ is a finite set of measurement points. We assume that $n_m \geq n_p$. The euclidian product and norm on Q and Z are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively, and the same notation is also used for the corresponding natural norms of matrices.

The values of the parameters are estimated from a given set of measurements $\hat{C} \in Z$ using a least squares approach such that we obtain a constrained optimization problem with the cost functional $J : V \rightarrow \mathbb{R}$:

$$\text{Minimize } J(u) := \frac{1}{2} \|C(u) - \hat{C}\|^2, \quad (1.5)$$

under the constraint (1.2). Throughout we assume the existence of a solution $(u, q) \in V \times Q_0$ of the problem (1.2,1.5). For an analysis of existence of solutions for parameter identification problems see, e.g., Banks & Kunisch [2], Kravaris & Seinfeld [14] and Litvinov [15].

The state equation is discretized by a conforming finite element Galerkin method defined on a family $\{\mathcal{T}_h\}_{h>0}$ of shape regular quasi-uniform meshes $\mathcal{T}_h = \{K\}$ consisting of closed *cells* K which are either triangles or quadrilaterals. The straight parts which make up the boundary ∂K of a cell K are called *faces*. The mesh parameter h is defined as a cell-wise constant function by setting $h|_K = h_K$ and h_K is the diameter of K . Usually we use the symbol h also for the maximal cell size, i.e.

$$h = \max_{K \in \mathcal{T}_h} h_K. \quad (1.6)$$

For convenience, we assume that $0 < h < 1$. On the mesh \mathcal{T}_h we define finite element spaces $V_h \subset V$ consisting of linear or bilinear shape functions, see e.g. Brenner & Scott [5] or Johnson [12]. The corresponding discrete state $u_h \in V_h$ and parameter $q_h \in Q_0$ are determined by:

$$\text{Minimize } J(u_h), \quad (1.7)$$

under the constraint

$$a(q_h)(u_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h. \quad (1.8)$$

Since Q is finite dimensional, the parameter q_h is determined in the same space Q .

The main purpose of this paper is to analyze the behavior of the error in parameters $\|q - q_h\|$ for h tending to zero. There is a number of publications, where a priori error estimates are derived for optimal control problems governed by partial differential equations, see Falk [7], Arada, Casas & Tröltzsch [1], Deckelnic & Hinze [6] and Gunzburger & Hou [11]. However, there are only few published results on this topic in the context of parameter identification problems, see Falk [8], Neittaanmäki & Tai [16] and Kärkkäinen [13].

In Vexler [21], an a priori error analysis for the case of V -stable observation operators $C_i(\cdot)$ is developed and optimal-order convergence is shown,

$$\|q - q_h\| = \mathcal{O}(h^2), \quad (1.9)$$

essentially under the assumption that $|C_i(u) - C_i(u_h)| \leq ch^2 \|u\|_{H^2}$. However, the generalization of this result to point-wise observations is not straightforward. For this case, we prove in this paper that, under certain regularity conditions,

$$\|q - q_h\| = \mathcal{O}(h^2 |\log(h)|^2). \quad (1.10)$$

The proof uses the technique for estimating discrete Green functions developed in Frehse & Rannacher [9]. A complementary result of a posteriori error analysis for parameter identification problems is given in Becker & Vexler [4].

To the authors knowledge, this is the first a priori error analysis for parameter identification problems with point-wise observations. The consideration of point-wise observations in determining discrete parameters seems very natural in view of practical measurement techniques; see [3] for applications in reactive flow analysis.

The paper is organized as follows: In the next section, we describe an algorithm for solving problem (1.7, 1.8). In Section 3, we present a paradigm for a priori error analysis for discretization of a class of optimization problems. Thereafter, in Section 4, we derive the announced error estimate using an L^∞ -stability result, which is proven in Section 5. In Section 6, we present a numerical example conforming the asymptotic sharpness of our error estimate. Possible extensions are addressed in the last section.

2. Optimization algorithm. In this section, we reformulate the problem under consideration as an unconstrained optimization problem and describe a solution algorithm for it. Throughout, we assume the coefficient matrix $A(q)$ to be positive definite at the solution of the problem. By virtue of the implicit function theorem for Banach spaces, this implies the existence of an open (bounded) set $Q_0 \subset Q$ containing the optimal parameter q , and of a continuously differentiable solution operator $S : Q_0 \rightarrow V$ satisfying:

$$a(q)(S(q), \phi) = (f, \phi) \quad \forall \phi \in V \quad \forall q \in Q_0. \quad (2.1)$$

Without loss of generality, we assume that $A(q)$ is positive definite for $q \in Q_0$, i.e. there exists $\gamma \in \mathbb{R}_+$, such that

$$p^* A(q) p \geq \gamma \|p\|^2, \quad \forall p \in Q \quad \forall q \in Q_0. \quad (2.2)$$

The convexity of the domain Ω implies the following regularity property (see e.g. Grisvard [10]):

$$S(q) \in V \cap H^2(\Omega) \quad \forall q \in Q_0. \quad (2.3)$$

We introduce the *reduced observation operator* $c : Q_0 \rightarrow Z$ by

$$c(q) := C(S(q)). \quad (2.4)$$

This allows us to reformulate the problem under consideration as an unconstrained optimization problem with the reduced cost functional $j : Q_0 \rightarrow \mathbb{R}$:

$$\text{Minimize } j(q) := \frac{1}{2} \|c(q) - \hat{C}\|^2, \quad q \in Q_0. \quad (2.5)$$

Denoting by $G = c'(q) \in \mathbb{R}^{n_p \times n_m}$ the Jacobian matrix of the reduced observation operator $c(\cdot)$, the first-order necessary optimality condition $j'(q) = 0$ for (2.5) reads

$$G^*(c(q) - \hat{C}) = 0, \quad (2.6)$$

where G^* denotes the transpose of G . The positive semi-definiteness of the Hessian matrix $H := \nabla^2 j(q)$ is the second-order necessary optimality condition. A solution q of problem (2.5) is called *stable*, if the sufficient optimality condition holds, i.e. if the Hessian H is positive definite. Throughout, we will assume the solution q to be stable. The stability of the solution is given, for instance, if the value of the cost functional $\|C(u) - \hat{C}\|$ is small enough and the matrix G has full rank n_p , see e.g. [21] for details.

Since by assumption the matrix coefficient $A(\cdot)$ is twice continuously differentiable, there holds

$$\sup_{\xi \in \bar{Q}_0} \|A(\xi)\|_{1,\infty} + \sup_{\xi \in \bar{Q}_0} \|A'_{q_j}(\xi)\|_{1,\infty} + \sup_{\xi \in \bar{Q}_0} \|A''_{q_j q_k}(\xi)\|_{1,\infty} < \infty, \quad (2.7)$$

where $\|B\|_{1,\infty} := \max_{i,j=1,2} \|B_{ij}\|_{1,\infty}$ for a matrix function $B = (B_{ij}) \in C^1(\bar{\Omega})^{2 \times 2}$.

In the following propositions, we give representations of the Jacobian G of $c(\cdot)$, the Hessian H of $j(\cdot)$, and the Hessian of $c_i(\cdot)$.

THEOREM 2.1. *Let the reduced observation operator $c(\cdot)$ and the reduced functional $j(\cdot)$ be defined as in (2.4) and (2.5), respectively. Then, the elements of the Jacobian of $c(\cdot)$ at some $q \in Q_0$ are given by*

$$G_{ij} = \frac{\partial c_i}{\partial q_j}(q) = C_i(w_j), \quad i = 1, \dots, n_m, \quad j = 1, \dots, n_p, \quad (2.8)$$

where $w_j \in V$ are the solutions of the problems

$$a(q)(w_j, \phi) = -(A'_{q_j}(q) \nabla u, \nabla \phi) \quad \forall \phi \in V, \quad (2.9)$$

with $u = S(q)$. The Hessian of $j(\cdot)$ can be expressed by

$$H = G^* G + M, \quad (2.10)$$

where the matrix $M \in \mathbb{R}^{n_p \times n_p}$ is given by

$$M = \sum_{i=1}^{n_m} c''_i(q)(c_i(q) - \hat{C}_i). \quad (2.11)$$

The Hessian of $c_i(q)$ is given by

$$\frac{\partial^2}{\partial q_j \partial q_k} c_i(q) = C_i(v_{jk}), \quad (2.12)$$

where the $v_{jk} \in V$ are the solutions of the problems

$$\begin{aligned} a(q)(v_{jk}, \phi) &= -(A'_{q_j}(q) \nabla w_k, \nabla \phi) - (A'_{q_k}(q) \nabla w_j, \nabla \phi) \\ &\quad - (A''_{q_j q_k}(q) \nabla u, \nabla \phi) \quad \forall \phi \in V, \end{aligned} \quad (2.13)$$

with w_j as defined in (2.9).

Proof. The derivation of the derivatives of $c(\cdot)$ is straightforward using the implicit function theorem and the chain rule. \square

In practice the Hessian H of $j(\cdot)$ is computed using the representation

$$M_{jk} = -(A'_{q_j}(q) \nabla w_k, \nabla z) - (A'_{q_k}(q) \nabla w_j, \nabla z) - (A''_{q_j q_k}(q) \nabla u, \nabla z), \quad (2.14)$$

with the function z determined by the dual equation

$$(A(q)\nabla\phi, \nabla z) = \langle C(u) - \bar{C}, C(\phi) \rangle. \quad (2.15)$$

Similar to the continuous case, we introduce a discrete solution operator $S_h : Q_0 \rightarrow V_h$ by the equation

$$a(q_h)(S_h(q_h), \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h, q_h \in Q_0. \quad (2.16)$$

As before, we turn the discrete problem (1.7, 1.8) into an unconstrained minimization problem:

$$\text{Minimize } j_h(q_h) := \frac{1}{2} \|c_h(q_h) - \hat{C}\|^2, \quad q_h \in Q_0, \quad (2.17)$$

where the discrete reduced observation operator c_h is defined by

$$c_h(q_h) = C(S_h(q_h)). \quad (2.18)$$

Denoting the corresponding Jacobian by $G_h = c'_h(q_h)$, the necessary optimality condition $j'_h(q_h) = 0$ reads

$$G_h^*(c_h(q_h) - \hat{C}) = 0. \quad (2.19)$$

The derivatives of the discrete observation operator c_h can be computed in an analogous way as in Theorem 2.1.

Problem (2.17) is solved iteratively starting with an initial guess q_h^0 and using the recursive setting $q_h^{k+1} = q_h^k + \delta q_h$. The update δq_h is obtained as the solution of the system of linear equations

$$H_k \delta q_h = G_h^*(\hat{C} - c_h(q_h^k)), \quad (2.20)$$

where $G_h = c'_h(q_h^k)$, and H_k is an appropriate symmetric approximation of the Hessian $\nabla^2 j_h(q_h^k)$. The most widely used choice of the matrix $H_k = G_h^* G_h$ leads us to the Gauß-Newton algorithm, see, e.g., Nocedal & Wright [19].

For one step of the Gauß-Newton algorithm the state equation and n_p tangent problems (2.9) have to be solved which involves the same linear operator but with different right-hand sides. Due to the small dimension n_p of the parameter space Q the solution of (2.20) is uncritical. For discussing other Newton type methods and trust-region techniques for globalization of the convergence in this context, see [21].

3. A paradigm for a priori error analysis. In this section we present a general approach to the error analysis of a class of optimization problems such as considered in this paper. The main result is stated in the following theorem. It is a variant of well-known perturbation theorems for differentiable mappings which is particularly tailored to the present situation. However, it seems easier to include the elementary proof than to search for the precise reference.

THEOREM 3.1. *Let $F, F_h : \mathbb{R}^n \rightarrow \mathbb{R}^n$, for a discretization parameter $h \in \mathbb{R}_+$, be continuously differentiable operators, and $x \in \mathbb{R}^n$ be a solution of $F(x) = 0$. Let the following conditions be fulfilled:*

(i) *The derivative $F'(x)$ is positive definite, i.e., there is a constant $\gamma > 0$, such that*

$$p^* F'(x) p \geq \gamma \|p\|^2, \quad p \in \mathbb{R}^n. \quad (3.1)$$

(ii) There is a neighborhood U of x and a positive number $L(h) \in \mathbb{R}_+$, such that

$$\|F'_h(\xi) - F'_h(\eta)\| \leq L(h)\|\xi - \eta\| \quad \forall \xi, \eta \in U. \quad (3.2)$$

(iii) With the h -dependent constant $L(h)$, there holds

$$\lim_{h \rightarrow 0} L(h)\|F(x) - F_h(x)\| = 0. \quad (3.3)$$

(iv) There holds

$$\lim_{h \rightarrow 0} \|F'(x) - F'_h(x)\| = 0. \quad (3.4)$$

Then, for h small enough, there exists $x_h \in U$, such that $F_h(x_h) = 0$, and $F'_h(x_h)$ is positive definite uniformly in h . Further, there holds the a priori error estimate

$$\|x - x_h\| \leq \frac{2}{\gamma}\|F(x) - F_h(x)\|. \quad (3.5)$$

Proof. Due to condition (iv), we can choose a positive number $h_1 \in \mathbb{R}_+$, such that for $h \leq h_1$ there holds

$$\|F'(x) - F'_h(x)\| \leq \frac{1}{4}\gamma. \quad (3.6)$$

Moreover, for $\rho = \rho(h) = \frac{\gamma}{kL(h)}$, with some $k \geq 4$ sufficiently large, there holds

$$B_\rho(x) = \{\xi \in \mathbb{R}^n, \|x - \xi\| \leq \rho\} \subset U. \quad (3.7)$$

For this choice, we obtain that, for $h \leq h_1$, $F'_h(\cdot)$ is positive definite on $B_\rho(x)$:

$$\begin{aligned} p^* F'_h(\xi)p &= p^* F'(x)p + p^*(F'_h(x) - F'(x))p + p^*(F'_h(\xi) - F'_h(x))p \\ &\geq \gamma\|p\|^2 - \|F'_h(x) - F'(x)\|\|p\|^2 - \|F'_h(\xi) - F'_h(x)\|\|p\|^2 \\ &\geq (\gamma - \frac{1}{4}\gamma - L(h)\rho)\|p\|^2 \geq \frac{1}{2}\gamma\|p\|^2. \end{aligned}$$

In a similar way, we conclude that, for $h \leq h_1$, $F'_h(\cdot)$ is also bounded on $B_\rho(x)$:

$$\|F'_h(\xi)\| \leq \beta := \|F'(x)\| + \frac{1}{2}\gamma.$$

Next, we prove, that there exists a unique $x_h \in B_\rho(x)$ with $F_h(x_h) = 0$. To this end, we define an operator $D_s : \mathbb{R}^n \rightarrow \mathbb{R}^n$, for $s \in \mathbb{R}_+$, by

$$D_s(\xi) = \xi - sF_h(\xi).$$

For a certain choice of s , we show that D_s is a contraction on $B_\rho(x)$ and use the Banach fixed point theorem. For $\xi \in B_\rho(x)$, $h \leq h_1$, and an arbitrary $p \in \mathbb{R}^n$, there holds

$$\begin{aligned} \|D'_s(\xi)p\|^2 &= \|p - sF'_h(\xi)p\|^2 = \|p\|^2 - 2sp^*F'_h(\xi)p + s^2\|F'_h(\xi)p\|^2 \\ &\leq (1 - s\gamma + s^2\beta^2)\|p\|^2. \end{aligned}$$

For the choice $s = \gamma(2\beta^2)^{-1}$, we obtain

$$\|D'_s(\xi)p\|^2 \leq \left(1 - \frac{\gamma^2}{4\beta^2}\right)\|p\|^2,$$

and consequently,

$$\|D'_s(\xi)\| \leq \left(1 - \frac{\gamma^2}{4\beta^2}\right)^{1/2} < 1.$$

Moreover, for arbitrary $\xi \in B_\rho(x)$, there holds:

$$\begin{aligned} \|x - D_s(\xi)\| &= \|D_s(x) - D_s(\xi) + sF_h(x)\| \\ &\leq \|D_s(x) - D_s(\xi)\| + s\|F_h(x) - F(x)\| \\ &\leq \|D'_s(\eta)\| \|x - \xi\| + s\|F_h(x) - F(x)\|, \end{aligned}$$

for a certain $\eta \in B_\rho$. Hence, the above estimate implies

$$\begin{aligned} \|x - D_s(\xi)\| &\leq \left(1 - \frac{\gamma^2}{4\beta^2}\right)^{1/2} \rho + s\|F_h(x) - F(x)\| \\ &= \rho \left\{ \left(1 - \frac{\gamma^2}{4\beta^2}\right)^{1/2} + s \frac{k}{\gamma} L(h) \|F_h(x) - F(x)\| \right\}. \end{aligned}$$

Due to condition (iii), there is a number $h_2 \in \mathbb{R}_+$, such that, for $h \leq h_2$, there holds

$$L(h) \|F_h(x) - F(x)\| \leq \frac{\gamma}{ks} \left\{ 1 - \left(1 - \frac{\gamma^2}{4\beta^2}\right)^{1/2} \right\}.$$

Hence, for $h < h_0 := \min\{h_1, h_2\}$,

$$\|x - D_s(\xi)\| \leq \rho,$$

and consequently $D_s(\xi) \in B_\rho(x)$. For $h \leq h_0$, by the Banach fixed point theorem, we obtain the existence of $x_h \in B_\rho(x)$ with $F_h(x_h) = 0$. By construction of $B_\rho(x)$ the derivative $F'_h(x_h)$ is positive definite with the h -independent constant $\frac{1}{2}\gamma$. This implies that, for a certain $\xi \in B_\rho(x)$,

$$(x - x_h)^*(F_h(x) - F_h(x_h)) = (x - x_h)^* F'_h(\xi)(x - x_h) \geq \frac{\gamma}{2} \|x - x_h\|^2.$$

Hence, using $F(x) = F_h(x_h) = 0$,

$$\begin{aligned} \|x - x_h\|^2 &\leq \frac{2}{\gamma} (x - x_h)^*(F_h(x) - F_h(x_h)) = \frac{2}{\gamma} (x - x_h)^*(F_h(x) - F(x)) \\ &\leq \frac{2}{\gamma} \|F_h(x) - F(x)\| \|x - x_h\|. \end{aligned}$$

This completes the proof. \square

4. A priori error estimation. In this section we apply the paradigm presented in Section 3 to the problem under consideration. We prove the following theorem.

THEOREM 4.1. *Let $q \in Q$ be a stable solution of (2.5). Then, for h small enough, there exist a stable solution $q_h \in Q$ of (2.17), and there holds the following a priori error estimate:*

$$\|q - q_h\| = \mathcal{O}(h^2 |\log(h)|^2). \quad (4.1)$$

On the basis of the estimate (4.1), we can also derive optimal-order estimates for the error $u - u_h$ in the corresponding states. However, since this would be a simple exercise using the arguments developed below, and since the optimal states are of only minor practical interest in parameter estimation problems, we do not state these estimates.

The proof of Theorem 4.1 is given by checking the conditions from Theorem 3.1 for the operators

$$F(\xi) := \nabla j(\xi) \quad F_h(\xi) := \nabla j_h(\xi).$$

The constant in (4.1) turns out to depend in a reciprocal way on the distance

$$\delta := \min_{i=1, \dots, n_m} \text{dist}(\xi_i, \Sigma)$$

of the set of measurement points ξ_i to the set Σ of corner points of $\partial\Omega$. Therefore, we will use generic constants c and c_δ , where c only depends on the domain Ω , the force f , and the characteristics of the mesh family $\{\mathcal{T}_h\}_h$, while c_δ may additionally depend on the distance δ like $c_\delta \approx \delta^{-1}$. Further, by $L^p(\Omega)$ and $W^{m,p}(\Omega)$, for $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, we denote the standard Lebesgue and Sobolev spaces and by $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$ the corresponding norms. The restriction of such a norm to a subset $\Omega' \subset \Omega$ is indicated by $\|\cdot\|_{m,p;\Omega'}$.

An important ingredient of the proof of Theorem 4.1 is the following L^∞ -stability theory. For $d > 0$, we define the subset $\Omega_d \subset \Omega$ by

$$\Omega_d := \{x \in \Omega, \text{dist}(x, \Sigma) > d\}.$$

THEOREM 4.2 (Stability theorem). *Let be given $q \in Q_0$, $\psi \in H_0^1(\Omega) \cap C(\bar{\Omega})$, and a matrix $B = B(x) \in W^{1,\infty}(\Omega)^{2 \times 2}$. Moreover, let $v_h \in V_h$ be a solution of*

$$a(q)(v_h, \phi_h) = (B \nabla \psi, \nabla \phi_h) \quad \forall \phi_h \in V_h. \quad (4.2)$$

Then, there hold the L^2 -stability estimate

$$\|v_h\|_2 + h \|\nabla v_h\|_2 \leq c \|B\|_{1,\infty} \{\|\psi\|_2 + h \|\nabla \psi\|_2\}, \quad (4.3)$$

and the local L^∞ -stability estimate

$$\|v_h\|_{\infty;\Omega_d} \leq c_d \|B\|_{1,\infty} \{|\log(h)| \|\psi\|_{\infty;\Omega_{d/2}} + \|\psi\|_2 + h \|\nabla \psi\|_2\}, \quad (4.4)$$

with a constant $c_d \approx d^{-1}$.

The L^2 estimate (4.3) is a standard result from finite element analysis, while the L^∞ estimate (4.4) follows by estimates of discrete Green functions such as developed in Frehse & Rannacher [9]. A similar L^∞ -stability result has been proven in Rannacher [20] in the time-dependent parabolic case. The proof is given in Section 5.

For the solution q of problem (2.5), we introduce $\bar{u}_h \in V_h$ determined by

$$a(q)(\bar{u}_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h. \quad (4.5)$$

Further, we define $w_{j,h} \in V_h$ and $v_{jk,h} \in V_h$, for $j, k = 1, 2, \dots, n_p$, as the solutions of the problems

$$a(q)(w_{j,h}, \phi_h) = -(A'_{q_j}(q) \nabla \bar{u}_h, \nabla \phi_h) \quad \forall \phi_h \in V_h, \quad (4.6)$$

and

$$a(q)(v_{jk,h}, \phi_h) = -(A'_{q_j}(q)\nabla w_{k,h}, \nabla \phi_h) - (A'_{q_k}(q)\nabla w_{j,h}, \nabla \phi_h) - (A''_{q_j q_k}(q)\nabla \bar{u}_h, \nabla \phi_h) \quad \forall \phi_h \in V_h, \quad (4.7)$$

respectively. The next lemma provides necessary estimates for the errors $u - \bar{u}_h$, $w_j - w_{j,h}$ and $v_{jk} - v_{jk,h}$. We recall the notation $\delta := \min_{i=1,\dots,n_m} \text{dist}(\xi_i, \Sigma)$.

LEMMA 4.3. *Under the above assumptions the following estimates hold:*

$$\|C(u - \bar{u}_h)\| \leq c_\delta h^2 |\log(h)|, \quad (4.8)$$

$$\|C(w_j - w_{j,h})\| \leq c_\delta h^2 |\log(h)|^2, \quad j = 1, 2, \dots, n_p, \quad (4.9)$$

$$\|C(v_{jk} - v_{jk,h})\| \leq c_\delta h^2 |\log(h)|^3, \quad j, k = 1, 2, \dots, n_p. \quad (4.10)$$

Proof. (i) The proof uses some a priori bounds for u and the auxiliary functions w_j, v_{jk} , $j, k = 1, \dots, n_p$, corresponding to arbitrary $q \in Q_0$. By our regularity assumptions, we have the following a priori estimates:

$$\|u\|_{2,2} \leq c, \quad (4.11)$$

$$\|w_j\|_{2,2} \leq c\|u\|_{2,2} \leq c, \quad (4.12)$$

$$\|v_{jk}\|_{2,2} \leq c \max\{\|w_j\|_{2,2}, \|w_k\|_{2,2}\} + c\|u\|_{2,2} \leq c, \quad (4.13)$$

where c is a generic constant depending only on the data of the problem. Further, by elliptic regularity theory, for $d > 0$, there hold the local a priori estimates

$$\|u\|_{C^{2+\alpha}(\bar{\Omega}_d)} \leq c_d \{ \|f\|_{C^\alpha(\bar{\Omega}_{d/2})} + \|u\|_{2,2} \} \leq c_d, \quad (4.14)$$

$$\|w_j\|_{C^{2+\alpha}(\bar{\Omega}_d)} \leq c_d \{ \|u\|_{C^{2+\alpha}(\bar{\Omega}_{d/2})} + \|u\|_{2,2} \} \leq c_d, \quad (4.15)$$

$$\|v_{jk}\|_{C^{2+\alpha}(\bar{\Omega}_d)} \leq c_d \{ \|w_j\|_{C^{2+\alpha}(\bar{\Omega}_{d/2})} + \|w_j\|_{2,2} \} \leq c_d, \quad (4.16)$$

with a generic constant $c_d \approx d^{-1}$. Here, $\|\cdot\|_{C^{2+\alpha}(\bar{\Omega}_d)}$ denotes the norm of the space $C^{2+\alpha}(\bar{\Omega}_d)$ of functions possessing Hölder continuous second derivatives on $\bar{\Omega}_d$.

(ii) By definition, \bar{u}_h is the Ritz projection of u corresponding to the energy form $a(q)(\cdot, \cdot)$, i.e.,

$$a(q)(\bar{u}_h, \phi_h) = a(q)(u, \phi_h) \quad \forall \phi_h \in V_h.$$

By the standard L^2 -error estimate for finite elements, there holds

$$\|u - \bar{u}_h\|_2 + h\|\nabla(u - \bar{u}_h)\|_2 \leq ch^2. \quad (4.17)$$

Further, let $i_h u \in V_h$ be the usual nodal interpolant of u . Then, applying the L^∞ -stability estimate (4.4) of Theorem 4.2 for the equation

$$a(q)(i_h u - \bar{u}_h, \phi_h) = a(q)(i_h u - u, \phi_h) \quad \forall \phi_h \in V_h,$$

yields the estimate

$$\|i_h u - \bar{u}_h\|_{\infty; \Omega_\delta} \leq c_\delta \{ |\log(h)| \|i_h u - u\|_{\infty; \Omega_{\delta/2}} + \|i_h u - u\|_2 + h\|\nabla(i_h u - u)\|_2 \}.$$

In view of the approximation properties of i_h and the a priori bounds (4.11) and (4.14), we conclude the error estimate

$$\|u - \bar{u}_h\|_{\infty; \Omega_\delta} \leq c_\delta h^2 |\log(h)|. \quad (4.18)$$

Since $\xi_i \in \bar{\Omega}_\delta$, we obtain the estimate (4.8).

(iii) For proving (4.9), we introduce an additional discrete variable $\bar{w}_{j,h}$ determined by the equation

$$a(q)(\bar{w}_{j,h}, \phi_h) = -(A'_{q_j}(q) \nabla u, \nabla \phi_h) \quad \forall \phi_h \in V_h.$$

The error $e = w_j - w_{j,h}$ is split like $e = e_1 + e_2$, with $e_1 = w_j - \bar{w}_{j,h}$ and $e_2 = \bar{w}_{j,h} - w_{j,h}$. For the Ritz-projection error e_1 , similar as before, there holds the L^2 -error estimate

$$\|e_1\|_2 + h \|\nabla e_1\|_2 \leq ch^2 \|w_j\|_{2,2} \leq ch^2,$$

and the pointwise error estimate

$$\|e_1\|_{\infty; \Omega_\delta} \leq c_\delta h^2 \{ |\log(h)| \|\nabla^2 w_j\|_{\infty; \Omega_{\delta/2}} + \|w_j\|_{2,2} \} \leq c_\delta h^2 |\log(h)|.$$

For $e_2 \in V_h$, we have

$$a(q)(e_2, \phi_h) = -(A'_{q_j}(q) \nabla(u - \bar{u}_h), \nabla \phi_h) \quad \forall \phi_h \in V_h.$$

Hence, the L^2 -stability estimate (4.3) of Theorem 4.2 and the estimate (4.17) imply

$$\|e_2\|_2 + h \|\nabla e_2\|_2 \leq c \{ \|u - \bar{u}_h\|_2 + h \|\nabla(u - \bar{u}_h)\|_2 \} \leq ch^2.$$

This shows that, for $j = 1, \dots, n_p$,

$$\|w_j - w_{j,h}\|_2 + h \|\nabla(w_j - w_{j,h})\|_2 \leq ch^2. \quad (4.19)$$

Further, the L^∞ -stability estimate (4.4) of Theorem 4.2 yields

$$\|e_2\|_{\infty; \Omega_\delta} \leq c_\delta \{ |\log(h)| \|u - \bar{u}_h\|_{\infty; \Omega_{\delta/2}} + \|u - \bar{u}_h\|_2 + \|\nabla(u - \bar{u}_h)\|_2 \}.$$

which, by (4.17) and (4.18), implies $\|e_2\|_{\infty; \Omega_\delta} \leq c_\delta |\log(h)|^2 h^2$. We obtain

$$\|w_j - w_{j,h}\|_{\infty; \Omega_\delta} \leq c_\delta |\log(h)|^2 h^2, \quad j = 1, \dots, n_p, \quad (4.20)$$

which implies the desired estimate (4.9).

(iv) The proof of (4.10) uses the same line of argument as before. Using the additional discrete variable $\bar{v}_{jk,h}$ determined by the equation

$$\begin{aligned} a(q)(\bar{v}_{jk,h}, \phi_h) &= -(A'_{q_j}(q) \nabla w_k, \nabla \phi_h) - (A'_{q_k}(q) \nabla w_j, \nabla \phi_h) \\ &\quad - (A''_{q_j q_k}(q) \nabla u, \nabla \phi_h) \quad \forall \phi_h \in V_h, \end{aligned}$$

the error $e = v_{jk} - v_{jk,h}$ is split like $e = e_1 + e_2$, with $e_1 = v_{jk} - \bar{v}_{jk,h}$ and $e_2 = \bar{v}_{jk,h} - v_{jk,h}$. For the Ritz-projection error e_1 , similar as before, we conclude the the pointwise error estimate

$$\|e_1\|_{\infty; \Omega_\delta} \leq c_\delta h^2 \{ |\log(h)| \|\nabla^2 v_{jk}\|_{\infty; \Omega_{\delta/2}} + \|v_{jk}\|_{2,2} \} \leq c_\delta h^2 |\log(h)|.$$

For $e_2 \in V_h$, we have

$$\begin{aligned} a(q)(e_2, \phi_h) &= -(A'_{q_j}(q)\nabla(w_k - w_{k,h}), \nabla\phi_h) - (A'_{q_k}(q)\nabla(w_j - w_{j,h}), \nabla\phi_h) \\ &\quad - (A''_{q_j q_k}(q)\nabla(u - \bar{u}_h), \nabla\phi_h) \quad \forall \phi_h \in V_h, \end{aligned}$$

and therefore, again by the L^∞ -stability estimate (4.4) of Theorem 4.2,

$$\begin{aligned} \|e_2\|_{\infty; \Omega_\delta} &\leq c_\delta |\log(h)| \left\{ \max_{j=1, \dots, n_p} \|w_j - w_{j,h}\|_{\infty; \Omega_{\delta/2}} + \|u - \bar{u}_h\|_{\infty; \Omega_{\delta/2}} \right\} \\ &\quad + c \max_{j=1, \dots, n_p} \left\{ \|w_j - w_{j,h}\|_2 + h \|\nabla(w_j - w_{j,h})\|_2 \right\} \\ &\quad + c \left\{ \|u - \bar{u}_h\|_2 + h \|\nabla(u - \bar{u}_h)\|_2 \right\}. \end{aligned}$$

Then, by the foregoing error estimates, we obtain $\|e_2\|_{\infty; \Omega_\delta} \leq c_\delta h^2 |\log(h)|^3$, and consequently,

$$\|v_{jk} - v_{jk,h}\|_{\infty; \Omega_\delta} \leq c_\delta |\log(h)|^3 h^2, \quad j, k = 1, \dots, n_p. \quad (4.21)$$

This eventually yields the desired estimated (4.10). \square

A direct application of Lemma 4.3 leads to the following result.

LEMMA 4.4. *Under the above assumptions, there holds*

$$\left| \frac{\partial}{\partial q_j} (j - j_h)(q) \right| \leq c_\delta h^2 |\log(h)|^2, \quad j = 1, 2, \dots, n_p, \quad (4.22)$$

$$\left| \frac{\partial^2}{\partial q_j \partial q_k} (j - j_h)(q) \right| \leq c_\delta h^2 |\log(h)|^3, \quad j, k = 1, 2, \dots, n_p. \quad (4.23)$$

Proof. We have the representation

$$\begin{aligned} \frac{\partial}{\partial q_j} (j - j_h)(q) &= \langle C(u) - \hat{C}, C(w_j) \rangle - \langle C(\bar{u}_h) - \hat{C}, C(w_{j,h}) \rangle \\ &= \langle C(u) - \hat{C}, C(w_j - w_{j,h}) \rangle + \langle C(u - \bar{u}_h), C(w_{j,h}) \rangle, \end{aligned}$$

from which we obtain

$$\left| \frac{\partial}{\partial q_j} (j - j_h)(q) \right| \leq \|C(u) - \hat{C}\| \|C(w_j - w_{j,h})\| + \|C(u - \bar{u}_h)\| \|C(w_{j,h})\|.$$

By the a priori bounds (4.11)-(4.13) and the Sobolev imbedding theorem, we see that

$$\|C(u)\| + \|C(w_j)\| + \|C(v_{jk})\| \leq c. \quad (4.24)$$

Combining this with the error estimate (4.9) implies $\|C(w_{j,h})\| \leq c$. Then, we can conclude the first estimate (4.22) from the error estimates of Lemma 4.3. To prove (4.23), we write

$$\begin{aligned} \frac{\partial^2}{\partial q_j \partial q_k} (j - j_h)(q) &= \langle C(w_j), C(w_k) \rangle + \langle C(u) - \hat{C}, C(v_{jk}) \rangle \\ &\quad - \langle C(w_{j,h}), C(w_{k,h}) \rangle - \langle C(\bar{u}_h) - \hat{C}, C(v_{jk,h}) \rangle \\ &= \langle C(w_j - w_{j,h}), C(w_k) \rangle + \langle C(w_{j,h}), C(w_k - w_{k,h}) \rangle \\ &\quad + \langle C(u - \bar{u}_h), C(v_{jk}) \rangle + \langle C(\bar{u}_h) - \hat{C}, C(v_{jk} - v_{jk,h}) \rangle. \end{aligned}$$

Using as before the bounds (4.24) and the error estimates of Lemma 4.3 completes the proof. \square

For the application of Theorem 3.1 it remains to check the Lipschitz condition (3.2). For two arbitrary parameter sets $\xi, \eta \in Q_0$, we set $u_\xi = S_h(\xi)$ and $u_\eta = S_h(\eta)$. Correspondingly, we define $w_{j,\xi}, w_{j,\eta} \in V_h$ and $v_{jk,\xi}, v_{jk,\eta} \in V_h$ similarly to $w_{j,h}$ and $v_{j,h}$ for $q = \xi$ and $q = \eta$, respectively.

LEMMA 4.5. *For $\xi, \eta \in Q_0$, there holds*

$$\|C(u_\xi - u_\eta)\| \leq c_\delta |\log(h)| \|\xi - \eta\|, \quad (4.25)$$

$$\|C(w_{j,\xi} - w_{j,\eta})\| \leq c_\delta |\log(h)|^2 \|\xi - \eta\|, \quad (4.26)$$

$$\|C(v_{jk,\xi} - v_{jk,\eta})\| \leq c_\delta |\log(h)|^3 \|\xi - \eta\|. \quad (4.27)$$

Proof. Due to the definition of u_ξ and u_η , we have

$$(A(\xi)\nabla(u_\xi - u_\eta), \nabla\phi_h) = -((A(\xi) - A(\eta))\nabla u_\eta, \nabla\phi_h) \quad \forall \phi_h \in V_h.$$

Using Theorem 4.2, with $d = \delta$, we obtain:

$$\|C(u_\xi - u_\eta)\| \leq c \| (A(\xi) - A(\eta)) \|_{1,\infty} \{ |\log(h)| \|u_\eta\|_{\infty, \Omega_{\delta/2}} + \|u_\eta\|_2 + h \|\nabla u_\eta\|_2 \}.$$

Since u_η is the Ritz projection of an H^2 function, all its norms occurring on the right-hand side are bounded independent of h and $\eta \in Q_0$ by standard estimates from finite element analysis. This implies (4.25) since

$$\| (A(\xi) - A(\eta)) \|_{1,\infty} \leq c \|\xi - \eta\|.$$

The estimates (4.26) and (4.27) are obtained in the similar way. \square

LEMMA 4.6. *For $\xi, \eta \in Q_0$, there holds:*

$$\left| \frac{\partial^2}{\partial q_j \partial q_k} j_h(\xi) - \frac{\partial^2}{\partial q_j \partial q_k} j_h(\eta) \right| \leq L(h) \|\xi - \eta\|, \quad (4.28)$$

where $L(h) = c_\delta |\log(h)|^3$.

Proof. We have

$$\begin{aligned} \frac{\partial^2}{\partial q_j \partial q_k} j_h(\xi) - \frac{\partial^2}{\partial q_j \partial q_k} j_h(\eta) &= \langle C(w_{j,\xi}), C(w_{k,\xi}) \rangle + \langle C(u_\xi) - \hat{C}, C(v_{jk,\xi}) \rangle \\ &\quad - \langle C(w_{j,\eta}), C(w_{k,\eta}) \rangle - \langle C(u_\eta) - \hat{C}, C(v_{jk,\eta}) \rangle \\ &= \langle C(w_{j,\xi} - w_{j,\eta}), C(w_{k,\xi}) \rangle + \langle C(w_{j,\eta}), C(w_{k,\xi} - w_{k,\eta}) \rangle \\ &\quad + \langle C(u_\xi - u_\eta), C(v_{jk,\xi}) \rangle + \langle C(u_\eta) - \hat{C}, C(v_{jk,\xi} - v_{jk,\eta}) \rangle, \end{aligned}$$

and, consequently,

$$\begin{aligned} \left| \frac{\partial^2}{\partial q_j \partial q_k} j_h(\xi) - \frac{\partial^2}{\partial q_j \partial q_k} j_h(\eta) \right| &\leq \|C(w_{j,\xi} - w_{j,\eta})\| \|C(w_{k,\xi})\| \\ &\quad + \|C(w_{j,\eta})\| \|C(w_{k,\xi} - w_{k,\eta})\| + \|C(u_\xi - u_\eta)\| \|C(v_{jk,\xi})\| \\ &\quad + \|C(u_\eta) - \hat{C}\| \|C(v_{jk,\xi} - v_{jk,\eta})\|. \end{aligned}$$

Now, the assertion follows by the estimates of Lemma 4.5 if we can bound the terms $C(u_\eta)$, $C(w_{j,\eta})$, and $C(v_{jk,\xi})$. This is achieved by using the bounds for $C(u)$, $C(w_j)$, and $C(v_{jk})$ in (4.24) together with the error estimates of Lemma 4.3. \square

To complete the proof of Theorem 4.1 we check the conditions of Theorem 3.1. Condition (3.1) is fulfilled due to the stability of the solution q of the problem (2.5). Condition (3.2) is shown in Lemma 4.6. Condition (3.3) is obtained by Lemma 4.4 and Lemma 4.6 using $\lim_{h \rightarrow 0} h^2 |\log(h)|^5 = 0$. Finally, condition (3.4) holds due to Lemma 4.4. Hence, the estimate (3.5) of Theorem 3.1 completes the proof.

5. Proof of Theorem 4.2. (i) We begin with the L^2 -stability estimate. Taking $\phi_h := v_h$ in (4.2), we obtain

$$\|\nabla v_h\|_2 \leq c \|B\|_{1,\infty} \|\nabla \psi\|_2. \quad (5.1)$$

To estimate $\|v_h\|_2$, we use the solution $z \in V \cap H^2(\Omega)$ of the auxiliary equation

$$a(q)(\phi, z) = (\phi, v_h) \|v_h\|_2^{-1} \quad \forall \phi \in V.$$

Then, taking $\phi := v_h$ and using the approximation properties of the interpolant $i_h z \in V_h$, we conclude that

$$\begin{aligned} \|v_h\|_2 &= a(q)(v_h, z) = a(q)(v_h, z - i_h z) + a(q)(v_h, i_h z) \\ &= a(q)(v_h, z - i_h z) + (B \nabla \psi, \nabla i_h z) \\ &= a(q)(v_h, z - i_h z) + (B \nabla \psi, \nabla(i_h z - z)) + (B \nabla \psi, \nabla z) \\ &\leq c \|\nabla v_h\|_2 \|\nabla(z - i_h z)\|_2 + \|B\|_{1,\infty} \|\nabla \psi\|_2 \|\nabla(z - i_h z)\|_2 \\ &\quad + \|B\|_{1,\infty} \|\psi\|_2 \|z\|_{2,2} \\ &\leq c \{h \|\nabla v_h\| + \|B\|_{1,\infty} h \|\nabla \psi\|_2 + \|B\|_{1,\infty} \|\psi\|_2\} \|z\|_{2,2}. \end{aligned}$$

Hence, observing (5.1) and the bound $\|z\|_{2,2} \leq c$, we obtain

$$\|v_h\|_2 \leq c \|B\|_{1,\infty} \{\|\psi\|_2 + h \|\nabla \psi\|_2\}. \quad (5.2)$$

(ii) Next, we prove the L^∞ -stability estimate. Let $a \in \Omega_\delta$ be an arbitrary point lying in a cell K . For any fixed h , there exists a cell-wise polynomial function δ_h with $\text{supp}(\delta_h) \subset K$, such that

$$(\phi_h, \delta_h) = \phi_h(a) \quad \forall \phi_h \in V_h.$$

The function δ_h plays the role of an approximate Dirac function. Correspondingly, we introduce a regularized Green function $g \in V \cap H^2(\Omega)$ by

$$(A(q)\nabla \phi, \nabla g) = (\delta_h, \phi) \quad \forall \phi \in V,$$

and the corresponding Ritz projection $g_h \in V_h$ by

$$(A(q)\nabla \phi_h, \nabla g_h) = (\delta_h, \phi_h) \quad \forall \phi_h \in V_h.$$

By $i_h : C(\bar{\Omega}) \rightarrow V_h$, we denote the usual (linear) operator of nodal interpolation for which the following cell-wise estimate is well known:

$$h_T^{-2} \|v - i_h v\|_{p;T} + h_T^{-1} \|\nabla(v - i_h v)\|_{p;T} + \|\nabla^2 i_h v\|'_{p;T} \leq c \|\nabla^2 v\|_{p;T}, \quad (5.3)$$

for $1 \leq p \leq \infty$, with constants c independent of h . For functions which are only cell-wise defined, we will use the norm $\|v\|_p := \sum_{K \in \mathcal{T}_h} \|v\|_{p;T}$. The following three lemmas provide the key estimates for the proof of the theorem.

LEMMA 5.1. *The following global L^2 estimates hold:*

$$\|g\|_2 + |\log(h)|^{-1/2} \|\nabla g\|_2 + h \|\nabla^2 g\|_2 \leq c, \quad (5.4)$$

$$h^{-1} \|g - g_h\|_2 + \|\nabla(g - g_h)\|_2 + h \|\nabla^2 g_h\|_2' \leq c. \quad (5.5)$$

Proof. The assertion follows by standard L^2 a priori and error estimates for g and $g - g_h$, respectively. We skip the details and refer to [9]. Note that $\|\nabla^2 g_h\|_2'$ vanishes for linear finite elements. In the case of bilinear elements, we estimate as follows:

$$\|\nabla^2 g_h\|_2' \leq \|\nabla^2(g_h - i_h g)\|_2' + \|\nabla^2(g - i_h g)\|_2' + \|\nabla^2 g\|_2.$$

For the first term, we obtain, using an inverse inequality,

$$\begin{aligned} \|\nabla^2(g_h - i_h g)\|_2' &\leq ch^{-1} \|\nabla(g_h - i_h g)\|_2 \\ &\leq ch^{-1} \{ \|\nabla(g - i_h g)\|_2 + \|\nabla(g - g_h)\|_2 \} \end{aligned}$$

and obtain by the interpolation estimate (5.3), with $p = 2$, and by the other estimates derived before:

$$\|\nabla^2 g_h\|_2' \leq c \{ h^{-1} \|\nabla(g - g_h)\|_2 + \|\nabla^2 g\|_2 \} \leq ch^{-1}.$$

This completes the proof. \square

LEMMA 5.2. *For sufficiently small $h \ll \delta$, the following local L^2 estimate holds:*

$$\|\nabla(g - g_h)\|_{2;\Omega \setminus \Omega_{\delta/2}} + h \|\nabla^2 g\|_{2;\Omega \setminus \Omega_{\delta/2}} \leq c_\delta h, \quad (5.6)$$

with a constant $c_\delta \approx \delta^{-1}$ but independent of h .

Proof. The assertion follows by standard local elliptic a priori estimates and by arguments from the local L^2 error analysis for finite elements as provided in Nitsche & Schatz [18]:

$$\begin{aligned} \|\nabla^2 g\|_{2;\Omega \setminus \Omega_{\delta/2}} &\leq c \|\Delta g\|_{2;\Omega \setminus \Omega_{3\delta/4}} + c_\delta \|g\|_2, \\ \|\nabla(g - g_h)\|_{2;\Omega \setminus \Omega_{\delta/2}} &\leq c \|\nabla(g - i_h g)\|_{2;\Omega \setminus \Omega_{3\delta/4}} + c_\delta \|g - g_h\|_2, \end{aligned}$$

with constants $c_\delta \approx \delta^{-1}$. Now, the assertion follows by the interpolation estimate (5.3) and the other estimates already proven. \square

LEMMA 5.3. *The following L^1 a priori and error estimates hold*

$$\|\nabla g\|_1 + \|\nabla^2 g\|_1' \leq c |\log(h)|, \quad (5.7)$$

$$\|\nabla(g - g_h)\|_1 + h \|\nabla^2(g - g_h)\|_1' \leq ch |\log(h)|, \quad (5.8)$$

with a constant c independent of h and δ .

Proof. The proof can be found in [9]. \square

For the point $a \in \Omega_d$, there holds

$$v_h(a) = (v_h, \delta_h) = (A(q)\nabla v_h, \nabla g_h) = (B\nabla\psi, \nabla g_h).$$

We employ a standard localization argument. Let $\omega \in C_0^\infty(\Omega)$ be a smooth function with the properties

$$0 \leq \omega \leq 1, \quad \omega|_{\Omega_{\delta/2}} \equiv 1, \quad \omega|_{\Omega \setminus \Omega_{\delta/4}} \equiv 0.$$

With this notation, we have

$$(B\nabla\psi, \nabla g_h) = (B\nabla(\omega\psi), \nabla g_h) + (B\nabla((1-\omega)\psi), \nabla g_h) =: \Sigma_1 + \Sigma_2.$$

First, we estimate the term Σ_1 . By integration by parts and observing that $\psi|_{\partial\Omega} = 0$, we obtain

$$(B\nabla(\omega\psi), \nabla g_h) = \sum_{K \in \mathcal{T}_h} \{(\omega\psi, -\nabla \cdot (B\nabla g_h))_T + (\omega\psi, n \cdot B\nabla g_h)_{\partial K \setminus \partial\Omega}\},$$

where n is the outward unit normal vector to ∂K . Let $[\nabla g_h]$ denote the jump of the gradient across the interior faces $\Gamma \subset \partial K$. Using this notation, we obtain

$$\Sigma_1 \leq \|\psi\|_{\infty; \Omega_{\delta/2}} \sum_{K \in \mathcal{T}_h} \{ \|\nabla \cdot (B\nabla g_h)\|_{1;K} + \frac{1}{2} \|n \cdot [B\nabla g_h]\|_{1; \partial K \setminus \partial\Omega} \}.$$

First, the estimates of Lemma 5.3 yield

$$\sum_{K \in \mathcal{T}_h} \|\nabla \cdot (B\nabla g_h)\|_{1;K} \leq c \|B\|_{1,\infty} \{ \|\nabla g_h\|_1 + \|\nabla^2 g_h\|'_1 \} \leq c \|B\|_{1,\infty} |\log(h)|.$$

Next, observing that $g \in H^2(\Omega)$ and therefore $[B\nabla g] = 0$, we obtain by a trace theorem:

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \|n \cdot [B\nabla g_h]\|_{1;K \setminus \partial\Omega} &= \sum_{K \in \mathcal{T}_h} \|n \cdot [B\nabla(g_h - g)]\|_{1; \partial K \setminus \partial\Omega} \\ &\leq c \|B\|_{1,\infty} \sum_{K \in \mathcal{T}_h} \{ h^{-1} \|\nabla(g - g_h)\|_{1;K} + \|\nabla^2(g - g_h)\|'_1 \}. \end{aligned}$$

Hence, collecting the foregoing estimates,

$$\Sigma_1 \leq c \|B\|_{1,\infty} \|\psi\|_{\infty; \Omega_{\delta/2}} \{ |\log(h)| + h^{-1} \|\nabla(g - g_h)\|_1 + \|\nabla^2(g - g_h)\|'_1 \}.$$

Again the estimates of Lemma 5.3, we obtain

$$\Sigma_1 \leq c \|B\|_{1,\infty} \|\psi\|_{\infty; \Omega_{\delta/2}} |\log(h)|. \quad (5.9)$$

For the term Σ_2 , we estimate as follows:

$$\begin{aligned} \Sigma_2 &= (B\nabla((1-\omega)\psi), \nabla(g_h - g)) + (B\nabla((1-\omega)\psi), \nabla g) \\ &\leq \|B\|_{1,\infty} \{ \|\nabla\psi\|_2 \|\nabla(g_h - g)\|_{2; \Omega \setminus \Omega_{\delta/2}} + c_\delta \|\psi\|_2 \|\nabla(g_h - g)\|_2 \} \\ &\quad + c \|B\|_{1,\infty} \|\psi\|_2 \{ \|\nabla^2 g\|_{2; \Omega \setminus \Omega_{\delta/2}} + \|\nabla g\|_{2; \Omega \setminus \Omega_{\delta/2}} \}. \end{aligned}$$

Then, by the L^2 estimates of Lemmas 5.1 and 5.2, it follows that

$$\Sigma_2 \leq c_\delta \|B\|_{1,\infty} \{ \|\psi\|_2 + h \|\nabla\psi\|_2 \}. \quad (5.10)$$

This completes the proof of the theorem.

6. Numerical results. In this section, we discuss a sample problem confirming the a priori error estimate of Theorem 4.1. The state equation is given by

$$\begin{aligned} -\nabla \cdot (A(q)\nabla u) &= 2 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{6.1}$$

where Ω is the unit square. The matrix $A(q)$ is a function of the parameter $q = (q_1, q_2) \in Q = \mathbb{R}^2$, given by

$$A(q) = \begin{pmatrix} q_1^2 & q_1 q_2 \\ q_1 q_2 & \exp(q_2) \end{pmatrix}.$$

The parameters are estimated from the measurements of the state variable at nine different points $\xi_i \in \Omega$, see Figure 6.1.

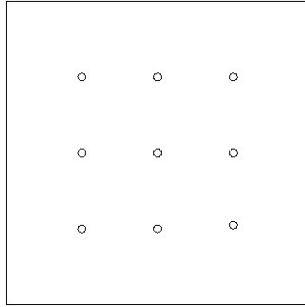


FIG. 6.1. The computational domain with measurement points marked by circles.

The vector of measurements \hat{C} is given by

$$\hat{C}_i = C_i(S(\hat{q}))(1 + \varepsilon_i) \quad i = 1, \dots, 9,$$

where the reference parameter is $\hat{q} = (5, 6)$ and $\varepsilon = (\varepsilon_i)$ describes the data perturbation. We consider two cases:

$$a) \varepsilon = 0, \quad b) \varepsilon \approx (0.12, -0.26, 0.29, -0.37, -0.49, 0.13, -0.04, -0.45, 0.20).$$

For case (a) the cost functional $J(u)$ vanishes in the optimal point. Case (b) is more realistic because of the “measurement errors” modelled by a randomly chosen ε . Moreover, in this case, in contrast to case (a) the solution q of the corresponding parameter identification problem and the reference parameter \hat{q} differ.

The parameter identification problem is discretized using bilinear finite elements on uniformly refined meshes. The results are listed in Table 6.1 and Table 6.2. For both cases the theoretically predicted orders of convergence are achieved.

7. Conclusions and Extensions. In this paper we have derived an a priori error estimate for the finite element discretization of an elliptic discrete parameter identification problem with pointwise measurements. The crucial point in our argument is the stability estimate of Theorem 4.2. The result of Theorem 4.1 can be extended as far as such a stability estimate is available. We list some possible directions of generalization.

1. *More general meshes:* For simplicity, we have assumed a quasi uniform mesh family $\{\mathcal{T}_h\}_h$. The analysis can be extended to locally refined meshes, provided that

TABLE 6.1

The error and the order of convergence with respect to the components of q without data perturbation.

N	$q_1 - q_{1,h}$	$q_2 - q_{2,h}$
81	5.955e-1	9.902e-4
289	1.407e-1	1.731e-4
1089	3.436e-2	4.343e-5
4225	8.509e-3	1.080e-5
16641	2.098e-3	2.668e-6
66049	4.993e-4	6.352e-7
order	2.05	2.04

TABLE 6.2

The error and the order of convergence with respect to the components of q with data perturbation.

N	$q_1 - q_{1,h}$	$q_2 - q_{2,h}$
81	2.059e-0	1.874e-2
289	5.172e-1	2.999e-3
1089	1.467e-1	8.341e-4
4225	3.771e-2	2.111e-4
16641	9.832e-3	5.640e-5
66049	2.350e-3	1.348e-5
order	2.01	1.98

the ratio of h_{\min} and $h = h_{\max}$ is polynomial, $h_{\min} \approx h^p$, with some $p \geq 1$. For such meshes the stability result of Theorem 4.2 holds true with $|\log(h_{\min})| \approx p|\log(h)|$.

2. *More general domains:* Our argument uses that the solution operator $S(\cdot)$ maps Q into $H_0^1(\Omega) \cap H^2(\Omega)$ which is guaranteed on smoothly bounded or convex domains. In the case of a domain with reentrant corners or edges this regularity property is lost. This lack of regularity of the solution can be compensated by an appropriate refinement of the mesh near the critical corner points or edges. The stability estimate of Theorem 4.2 also holds in this situation. This is true in two as well as in three dimensions. This will be shown in a forthcoming paper.

3. *Higher order approximation:* The result of Theorem 4.1 can be also extended to the case of higher-order finite elements, similar to the analysis of Nitsche [17]. In this case the logarithmic factor $|\log(h)|$ can be dropped in the stability Theorem 4.2.

4. *More general equations:* Theorem 4.1 can also be extended to more general elliptic equations or systems of the form

$$-\nabla \cdot (A(q)\nabla u + b_1(q)u) + b_2(q)u = f,$$

with parameter depended coefficients $A(q)$, $b_1(q)$, $b_2(q)$.

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