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Adaptive FEM for eigenvalue problems with application in hydrodynamic stability analysis

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Dedicated to Yuri Kuznetsov on the occasion of his 60th birthday.

Abstract — We present an adaptive finite element method for the solution of eigenvalue problems associated with the linearized stability analysis of non-linear operators in the context of hydrodynamic stability theory. The general framework is the Dual Weighted Residual (DWR) method for local mesh adaptation which is driven by residual-based and sensitivity-controlled a posteriori information. The basic idea is to embed the eigenvalue approximation into the general framework of Galerkin methods for nonlinear variational equations for which the DWR method is already well developed. The evaluation of these error representations results in *a posteriori* error bounds for approximate eigenvalues reflecting the errors by discretization of the eigenvalue problem as well as those by linearization about an only approximately known base solution. From these error estimates local error indicators are derived by which economical meshes can be constructed. The practical features of this method are illustrated by several examples.

Keywords: Eigenvalue problems, finite elements, a posteriori error estimation, mesh adaptivity, DWR method, hydrodynamic stability

1. INTRODUCTION

We consider the Galerkin finite element approximation of eigenvalue problems arising in the linearized stability theory of nonlinear variational equations particularly in hydrodynamic stability theory. Some parts of the underlying theory will be developed within a more abstract setting suggesting the application to other kinds of nonlinear problems.

On a bounded domain $\Omega \subset \mathbb{R}^d$ with $d = 2$ or $d = 3$, we consider elliptic eigenvalue problems of increasing complexity, such as symmetric diffusion problems

$$-\Delta v = \lambda v, \quad \text{in } \Omega, \quad v_{|\partial\Omega} = 0, \tag{1.1}$$

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nonsymmetric convection-diffusion problems

$$-\Delta v + b\cdot\nabla v = \lambda v, \quad \text{in } \Omega, \quad v_{|\partial\Omega} = 0, \tag{1.2}$$

and eigenvalue problems in hydrodynamic stability theory,

$$\begin{aligned} &-\nu\Delta v + \hat{v}\cdot\nabla v + v\cdot\nabla\hat{v} + \nabla q = \lambda v, \ \nabla\cdot v = 0, \quad \text{in } \Omega, \\ &v|_{\Gamma_{\text{rigid}}} = 0, \ v|_{\Gamma_{\text{in}}} = 0, \ \nu\partial_n v - pn|_{\Gamma_{\text{out}}} = 0. \end{aligned} \tag{1.3}$$

Here, $\hat{u} := \{\hat{v}, \hat{p}\}$ is a stationary 'base flow', that is a stationary solution of the Navier-Stokes equations

$$\begin{aligned} &-\nu\Delta v + v\cdot\nabla v + \nabla p = f, \ \nabla\cdot v = 0, \quad \text{in } \Omega, \\ &v|_{\Gamma_{\text{rigid}}} = 0, \ v|_{\Gamma_{\text{in}}} = v^{\text{in}}, \ \nu\partial_n v - pn|_{\Gamma_{\text{out}}} = 0, \end{aligned} \tag{1.4}$$

where v is the velocity vector field of the flow, p is its hydrostatic pressure, ν is the kinematic viscosity (density $\rho \equiv 1$), and f is a prescribed volume force. The goal is to investigate the stability of the base flow $\{\hat{v}, \hat{p}\}$ under small perturbations, which leads us to consider the eigenvalue problem (1.3). If an eigenvalue of (1.3) has $\operatorname{Re}\lambda < 0$, the base flow is unstable, otherwise it is said to by 'linearly stable'. That means that the solution of the linearized nonstationary perturbation problem

$$\partial_t w - \nu\Delta w + \hat{v}\cdot\nabla w + w\cdot\nabla\hat{v} + \nabla q = 0, \quad \nabla\cdot w = 0, \quad \text{in } \Omega, \tag{1.5}$$

corresponding to some initial perturbation $w_{|t=0} = w^0$ satisfies a bound of the form

$$\sup_{t\geqslant 0} \|w(t)\| \leqslant A\|w^0\|, \tag{1.6}$$

for some constant $A \geqslant 1$, where $\|\cdot\|$ denotes the L^2-norm over Ω and $(\cdot,\cdot)$ the corresponding inner product. However, *linear* stability does not guarantee full *nonlinear* stability due to effects caused by the non-normality of problem (1.3) making the constant A large. We will address this point in more detail below.

The finite element discretization of these eigenvalue problems is based on variational formulations. It uses finite element spaces V_h consisting of piecewise polynomial functions on certain decompositions $\mathbb{T}_h$ of the domain Ω into cells $T \in \mathbb{T}_h$ (triangle, quadrilateral, etc.) with diameter $h_T := \operatorname{diam}(T)$. Our primal goal is to derive computable *a posteriori* estimates for the error $\lambda - \lambda_h$ in terms of the 'cell residuals' of the computed approximations $\hat{v}_h$ and $\{v_h, \lambda_h\}$. For symmetric eigenvalue problems of the kind (1.1) *a posteriori* error estimates of this type have been derived by Nystedt [29], Larson [28], and Verfürth [35]. Their proofs largely exploit symmetry and also require the H^2-regularity of the problem which excludes domains with reentrant corners. In Heuveline & Rannacher [17] nonsymmetric eigenvalue problems of the kind (1.2) have been treated by employing duality techniques from optimal control theory. In this approach the central idea is to simultaneously consider the approximation of the the 'primal' eigenvalue problem $\mathscr{A}u = \lambda u$ and

its associated 'dual' analogue $\mathscr{A}^* u^* = \lambda^* u^*$. The combined problem is embedded into the general optimal-control framework of Galerkin approximations of nonlinear variational equations developed in Becker & Rannacher [6,7]. In this paper, we extend this approach to the case of nonsymmetric stability eigenvalue problems with special emphasis on non-normality effects and coefficient perturbation. The result is an *a posteriori* error estimate of the form

$$|\lambda - \lambda_h| \approx \sum_{T \in \mathbb{T}_h} h_T^2 \left\{ \hat{\rho}_T(\hat{u}_h)\hat{\rho}_T^*(\hat{u}_h^*) + \rho_T(u_h,\lambda_h)\,\rho_T^*(u_h^*,\lambda_h^*) \right\}, \tag{1.7}$$

involving the cell residuals $\hat{\rho}_T$, $\hat{\rho}_T^*$, and ρ_T, ρ_T^* of the computed primal and dual base solutions $\hat{u}_h$, $\hat{u}_h^*$, and eigenpairs $\{u_h,\lambda_h\}$, $\{u_h^*,\lambda_h^*\}$, respectively. This error estimate accomplishes simultaneous control of the error in the linearization, $\hat{v} - \hat{v}_h$, and the error in the resulting eigenvalues, $\lambda - \lambda_h$. The cellwise error indicators can then guide the mesh refinement process. Further, we devise a simple computational criterion based on the inner products (v_h, v_h^*) which can detect possible instability caused by non-normality effects. The practical features of this method will be illustrated for some model situations of transition from stability to instability including 2D channel flow around a cylinder.

We note that the approach of simultaneously considering the primal and dual eigenvalue problems is also crucial for constructing efficient multigrid solvers for the discrete problems. This has been successfully exploited for nonsymmetric elliptic eigenvalue problems in Heuveline & Bertsch [16].

The *a priori* error analysis for the nonsymmetric eigenvalue problem is well developed in the literature, Bramble & Osborn [9], Osborn [30], Kolata [25], Babuska & Osborn [1], and the literature cited therein). Particularly to mention is Osborn [30], where the eigenvalue problem of the linearized Navier-Stokes equations is considered, though neglecting the additional error due to the approximative linearization. These studies usually employ the heavy machinery of resolvent integral calculus as described in Kato [26]. We will see that the *a posteriori* error analysis only needs arguments from elementary calculus since it is based on the assumption that the approximation is sufficiently accurate on the considered meshes. The justification of this assumption is supplied by the *a priori* error analysis.

The further contents of this paper are as follows. In Sections 2, we discuss the special features of the stability eigenvalue problem of the Navier-Stokes equations particularly the consequences arising from its non-normality. Section 3 recalls the details of the finite element approximation of problems (1.4) and (1.3) from Rannacher [31] and Braack & Rannacher [8]. Then, Section 4 outlines the general framework of elliptic eigenvalue problems and their finite element approximation developed in Heuveline & Rannacher [17]. Section 5 presents the application of the abstract theory to the stability eigenvalue problem of the linearized Navier-Stokes equations. In Section 6, we discuss the practical aspects of a posteriori error estimation and mesh adaptation. Finally, Section 7 illustrates the theoretical results for some model eigenvalue problems, related to a scalar convection-diffusion equation, the Burgers equation, and the Navier-Stokes equations exhibiting instability.

2. THE NAVIER-STOKES STABILITY EIGENVALUE PROBLEM

The starting point is a variational formulation of the Navier-Stokes problem. We use the common notation $L^2(\Omega)$ and $H_0^m(\Gamma;\Omega) \subset H^m(\Omega)$ for the Lebesgue and Sobolev spaces on Ω, with the corresponding norms denoted by $\|\cdot\|$ and $\|\cdot\|_m$, respectively. Spaces of $\mathbb{R}^d$-valued functions $v = (v_1, \ldots, v_d)$ are denoted by boldface-type, but no distinction is made in the notation of norms and inner products; thus $\mathbf{H}_0^1(\Gamma;\Omega) = H_0^1(\Gamma;\Omega)^d$ has norm $\|v\|_1 = (\sum_{i=1}^d \|v_i\|^2)^{1/2}$, etc. All other notation are self-evident, e.g., $\partial_t u = \partial u/\partial t$ and $\partial_n v = n \cdot \nabla v$, where n is an outer normal unit vector. We assume $\partial\Omega = \Gamma_{\rm in} \cup \Gamma_{\rm out} \cup \Gamma_{\rm rigid}$, where $\Gamma_{\rm in}$, $\Gamma_{\rm out}$, and $\Gamma_{\rm rigid}$ denote the inlet, the outlet and the rigid part of the boundary, respectively. We introduce the abbreviation

$$L := L^2(\Omega), \quad \hat{\mathbf{H}} := \mathbf{H}^1(\Omega), \quad \mathbf{H} := \{v \in \mathbf{H}^1(\Omega),\, v_{|\Gamma_{\rm in}\cup\Gamma_{\rm rigid}} = 0\},$$

and set

$$\hat{\mathbf{V}} := \hat{\mathbf{H}} \times L, \quad \mathbf{V} := \mathbf{H} \times L \subset \hat{\mathbf{V}}.$$

In the case $\Gamma_{\rm out} = \emptyset$, we use $L := L_0^2(\Omega) = \{q \in L^2(\Omega) : (q,1) = 0\}$. For pairs $u = \{v, p\}$ and $\varphi = \{\varphi^v, \varphi^p\} \in \hat{\mathbf{V}}$, we define the semilinear form

$$a(u;\varphi) := \nu(\nabla v, \nabla\varphi^v) + (v \cdot \nabla v, \varphi^v) - (p, \nabla\cdot\varphi^v) + (\varphi^p, \nabla\cdot v),$$

and the right-hand side $F(\varphi) := (f, \varphi^v)$. Then, with a solenoidal extension $\hat{v}^{\rm in} \in \hat{\mathbf{V}}$ of the inflow data $v^{\rm in}$, we consider a solution $\hat{u} = \{\hat{v}, \hat{p}\} \in \mathbf{V} + \{\hat{v}^{\rm in}, 0\}$ of the variational problem

$$a(\hat{u};\varphi) = F(\varphi) \quad \forall \varphi \in \mathbf{V}, \tag{2.1}$$

or $\mathscr{A}(\hat{u}) = F$ in operator notation. In the following, we do not need H^2-regularity of this problem therefore allowing for general domains which may even be non-convex polygonal or polyhedral.

It is convenient to introduce spaces of solenoidal functions in order to formally eliminate the pressure from the discussion. This is justified since for a solution $u = \{v, p\} \in \mathbf{V} = \mathbf{H} \times L$ of the variational Navier-Stokes problem (2.1), the pressure is uniquely determined in L by the corresponding velocity $v \in \mathbf{H}$ through the equation

$$(p, \nabla\cdot\varphi) = \nu(\nabla v, \nabla\varphi) + (v \cdot \nabla v, \varphi) - (f, \varphi) \quad \forall \varphi \in \mathbf{H}. \tag{2.2}$$

This is a consequence of the well-known 'inf-sup' inequality (see Girault & Raviart [15] or Galdi [14])

$$\inf_{p\in L} \left(\sup_{v\in\mathbf{H}} \frac{(p, \nabla\cdot\varphi)}{\|\varphi\|_1 \, \|p\|} \right) \geqslant \gamma_0 > 0. \tag{2.3}$$

In the following discussion, we will work with the spaces

$$\mathbf{J}_1 := \{v \in \mathbf{H}, \nabla\cdot v = 0\}, \qquad \mathbf{J}_0 := \overline{\mathbf{J}_1}^{\|\cdot\|}.$$

However, in the context of discretization by the finite element method, we will have to come back to the velocity-pressure spaces $\mathbf{V} = \mathbf{H} \times L$, since in practice one usually does not work with exactly solenoidal trial functions.

We assume that the base solution $\hat{v} \in \mathbf{J}_1$ is (locally) unique, that is the Fréchet derivative $a'(\hat{v};\cdot,\cdot)$ taken at $\hat{v}$ is regular on $\mathbf{V}$ and with some $\gamma \in \mathbb{R}_+$, there holds

$$\inf_{v \in \mathbf{J}_1} \left(\sup_{\varphi \in \mathbf{J}_1} \frac{a'(\hat{v}; v, \varphi)}{\|v\|_1 \|\varphi\|_1} \right) \geqslant \gamma > 0. \tag{2.4}$$

Then, the a priori error analysis guarantees that $\hat{v}$ can be approximated by the finite element scheme. The variational formulation of the corresponding eigenvalue problem uses the derivative form

$$a'(\hat{v}; \psi, \varphi) := \nu(\nabla\psi, \nabla\varphi) + (\hat{v} \cdot \nabla\psi, \varphi) + (\psi \cdot \nabla\hat{v}, \varphi), \quad \varphi, \psi \in \mathbf{J}_1.$$

Then, the eigenvalue problem associated to the base solution $\hat{v}$ determines $\lambda \in \mathbb{C}$ and $v \in \mathbf{J}_1 \backslash \{0\}$ satisfying

$$a'(\hat{v}; v, \varphi) = \lambda\,(v, \varphi) \quad \forall \varphi \in \mathbf{J}_1, \tag{2.5}$$

or $\mathscr{A}'(\hat{v})v = \lambda v$ in operator notation. Since the domain Ω is bounded, the classical Riesz-Schauder theory implies that this eigenvalue problem possesses a countably infinite set $\Sigma(\mathscr{A}'(\hat{v})) := \{\lambda_i\}_{i=1}^{\infty} \subset \mathbb{C}$ of isolated eigenvalues with finite (algebraic) multiplicities which have no finite accumulation points. The difference between the algebraic and geometric multiplicity of an eigenvalue λ, its so-called 'defect', is denoted by $\alpha \in \mathbb{N}_0$ and corresponds to the largest integer such that

$$N((\mathscr{A}'(\hat{v}) - \lambda\mathscr{J})^{\alpha+1}) \neq N((\mathscr{A}'(\hat{v}) - \lambda\mathscr{J})^{\alpha}).$$

The 'primal' (right) eigenfunctions are normalized by $\|v\| = 1$. For the following discussion, we assume that the eigenvalue of interest, λ, has *geometric multiplicity one*. The case of higher geometric multiplicity requires some obvious changes. Associated to the primal eigenfunction $v \in \mathbf{J}_1$, there is a 'dual' (left) eigenfunction $v^* \in \mathbf{J}_1 \backslash \{0\}$ corresponding to λ, that is determined by the 'dual' eigenvalue problem

$$a'(\hat{v}; \varphi, v^*) = \lambda\,(\varphi, v^*) \quad \forall \varphi \in \mathbf{J}_1, \tag{2.6}$$

or $\mathscr{A}'(\hat{v})^* v^* = \lambda^* v^*$ in operator notation. The dual eigenfunction may be normalized by $(v, v^*) = 1$. If $(v, v^*) = 0$, then (and only then) the problem

$$a'(\hat{v}; v^1, \varphi) - \lambda\,(v^1, \varphi) = (u, \varphi) \quad \forall \varphi \in \mathbf{J}_1, \tag{2.7}$$

possesses a solution $u^1 \in \mathbf{J}_1$, a 'generalized eigenfunction', satisfying $(v^1, v) = 0$. In this case the eigenvalue λ has defect $\alpha \geqslant 1$. If also $(v^1, v^*) = 0$, then we obtain a generalized eigenfunction $u^2 \in \mathbf{J}_1$ of second order as a solution of

$$a'(\hat{v}; v^2, \varphi) - \lambda\,(v^2, \varphi) = (v^1, \varphi) \quad \forall \varphi \in \mathbf{J}_1, \tag{2.8}$$

satisfying $(v^2, v) = 0$. We can continue this construction of generalized eigenfunctions $v^1, v^2, \ldots$ of increasing order until $(v^\alpha, v^*) \neq 0$ occurs. Then, the defect of λ is exactly α. The result (1.7) has been derived in [17] under the restrictive assumption that the eigenvalue λ to be approximated is simple and consequently has defect $\alpha = 0$. Here, the *geometric* simplicity is not essential but so is the exclusion of higher *algebraic* multiplicity. The following analysis will also include the case $\alpha > 0$. For the details, we refer to Heuveline & Rannacher [18]. The effect of degeneracy on the numerical approximation of the Navier-Stokes equations has been addressed in Johnson & Rannacher [22], Johnson, Rannacher& Boman [23].

2.1. The effect of non-normality

In general, under discretization an eigenvalue λ with algebraic multiplicity $m \geqslant 2$ will split into a group of m simple eigenvalues $\{\lambda_h^{(i)}, i = 1, \ldots, m\}$ which may be considered as the approximation to λ. Only in special cases, for instance when the discretization preserves certain symmetry properties of the continuous problem, some of the discrete eigenvalues may have geometric multiplicity greater than one. Further these eigenvalues will usually be nondegenerate because of asymmetries in the discretization. The existence of an eigenvalue with $\mathrm{Re}\lambda < 0$ inevitably causes dynamic instability of the base flow $\hat{v}$, i.e., arbitrarily small perturbations may grow without bound. This is implied by the growth property

$$\|S(t)\| \approx A t^\alpha e^{-\mathrm{Re}\lambda\, t} \to \infty \quad (t \to \infty) \tag{2.9}$$

of the solution operator $S : \mathbf{J}_0 \to \mathbf{J}_0$ for the linearized perturbation equation

$$(\partial_t w, \varphi) + a'(\hat{v}; w, \varphi) = 0 \quad \forall \varphi \in \mathbf{J}_1. \tag{2.10}$$

Therefore, we are mainly interested in the case $\Sigma(\mathscr{A}'(\hat{v})) \subset \mathbb{C}_+ := \{z \in \mathbb{C}, \mathrm{Re}z > 0\}$ and want to compute the most 'critical' eigenvalues, that is those λ with minimal real part $\mathrm{Re}\lambda > 0$. The crucial question is how to detect *a posteriori* whether the growth factor At^α in the estimate (2.9) may become critical or not.

To this end, we may employ the following simple indicator: For any small $h > 0$, let λ_h be one of the approximating eigenvalues of $\mathscr{A}_h'(\hat{v})$ with corresponding right and left eigenvectors v_h and v_h^*, satisfying $\|v_h\| = 1$ and $(v_h, v_h^*) = 1$. By theory, there are associated (normalized) right and left eigenvectors (depending on h), say v and v^*, corresponding to the eigenvalue λ of $\mathscr{A}'(\hat{v})$, such that $\|v_h - v\| \to 0$ and $\|v_h^* - v^*\| \to 0$, as $h \to 0$. If the limit eigenvalue λ is deficient, then there must be an algebraic eigenvector w associated to v, such that

$$\mathscr{A}'(\hat{v})w - \lambda w = v.$$

The condition for the existence of such an algebraic eigenvector is that $(v, v^*) = 0$. Hence, in view of the normalization $\|v_h\| = (v_h, v_h^*) = 1$, the blow-up

$$\|v_h^*\| \to \infty \quad (h \to 0), \tag{2.11}$$

with a certain rate, can be used as an indicator for $\alpha(\lambda) > 0$. In this case the behavior (2.9) of the solution operator $S(t)$ implies

$$\sup_{t>0} \|S(t)\| \geqslant \frac{\alpha}{|\mathrm{Re}\lambda|}. \tag{2.12}$$

Consequently, for $0 < \mathrm{Re}\lambda \ll 1$ the amplification constant A in the stability estimate (1.6) becomes very large, i.e., small perturbations may initially be amplified to an extent such that nonlinear instability occurs.

However, a similar effect is also possible for non-deficient eigenvalues λ. This is related to the concept of the 'pseudo-spectrum' described in Trefethen et al. [33] (see also Landahl [27]). For $\varepsilon \in \mathbb{R}_+$ the ε-pseudo-spectrum $\sigma_\varepsilon \subset \mathbb{C}$ of the operator $\mathscr{A}'(\hat{v})$ is defined by

$$\sigma_\varepsilon := \big\{ z \in \mathbb{C},\ \|(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}\| \geqslant \varepsilon^{-1} \big\}.$$

In the following, we recall a result on the possible size of the amplification constant A which in the finite dimensional case may be obtained from the 'Kreiss matrix theorem' (see [34] and the references cited therein). For completeness, we supply a simple proof by energy arguments.

Proposition 2.1. *Let $z \in \mathbb{C}$ be a regular point of the operator $\mathscr{A}'(\hat{v})$ with $\mathrm{Re}z < 0$. Then, for the solution operator $S(t) : \mathbf{J}_0 \to \mathbf{J}_0$ of the linear perturbation equation (*2.10*), there holds*

$$\sup_{t\geqslant 0} \|S(t)\| \geqslant |\mathrm{Re}z|\, \|(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}\|. \tag{2.13}$$

Proof. Notice that, for $z \notin \sigma(\mathscr{A}'(\hat{v}))$, the resolvent $(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}$ is well defined as a bounded operator in $\mathbf{J}_0$, and there holds

$$\|(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}\|^{-1} = \inf_{v\in\mathbf{J}_1} \left(\sup_{\varphi\in\mathbf{J}_1} \frac{|a'(\hat{v}; v, \varphi) - z\,(v, \varphi)|}{\|v\|\,\|\varphi\|} \right). \tag{2.14}$$

We rewrite the perturbation equation (2.10) in the form

$$(\partial_t v, \varphi) + z(v, \varphi) + a'(\hat{v}; v, \varphi) - z(v, \varphi) = 0,$$

and multiply by e^{tz}, to obtain

$$\frac{d}{dt}\big[e^{tz}(v, \varphi)\big] + e^{tz}\big(a'(\hat{v}; v, \varphi) - z(v, \varphi)\big) = 0.$$

Next, integrating this with respect to t, we conclude

$$|(v^0, \varphi)| \leqslant e^{t\mathrm{Re}z}|(v(t), \varphi)| + \sup_{\psi\in\mathbf{J}_1} \frac{a'(\hat{v}; \psi, \varphi) - z(\psi, \varphi)}{\|\psi\|} \int_0^t e^{s\mathrm{Re}z}\|v\|\, ds.$$

Taking $\varphi = v^0$, and observing $\mathrm{Re}z < 0$, we conclude that

$$\|v^0\| \leqslant \big(e^{t\mathrm{Re}z} + \beta_z(v^0)|\mathrm{Re}z|^{-1}\big) \max_{[0,t]} \|v\|,$$

with the notation

$$\beta_z(v^0) := \sup_{\psi \in \mathbf{J}_1} \frac{a'(\hat{v};\psi,v^0) - z(\psi,v^0)}{\|\psi\|}.$$

Hence, recalling that $v(t) = S(t)v^0$,

$$1 \leqslant \Big(e^{t\mathrm{Re}z} + \frac{\beta_z(v^0)}{|\mathrm{Re}z|}\Big) \max_{[0,t]} \|S\|.$$

Since $\inf_{v^0 \in \mathbf{J}_1} \beta_z(v^0) = \|(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}\|^{-1}$, we conclude that

$$|\mathrm{Re}z|\,\|(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}\| \leqslant \big(e^{t\mathrm{Re}z}|\mathrm{Re}z|\,\|(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}\| + 1\big) \max_{[0,t]} \|S\|,$$

for all $t \geqslant 0$. From this, the asserted estimate follows. Because, assuming the contrary, we would have

$$\|(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}\| < \big(e^{t\mathrm{Re}z}|\mathrm{Re}z|\,\|(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}\| + 1\big)\|(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}\|,$$

which, since $\mathrm{Re}z < 0$, would result in a contradiction for sufficiently large t. □

The next proposition relates the size of the resolvent norm $\|(\mathscr{A}'(\hat{v}) - z\mathscr{I})^{-1}\|$ to easily computable quantities in terms of the eigenvalues and eigenfunctions of the operator $\mathscr{A}'(\hat{v})$. For the proof, we refer to Heuveline & Rannacher [18].

Proposition 2.2. *Let $\lambda \in \mathbb{C}$ be a non-deficient eigenvalue of the operator $\mathscr{A}'(\hat{v})$ with corresponding primal and dual eigenvectors $v, v^* \in \mathbf{J}_1$ normalized by $\|v\| = (v, v^*) = 1$. Then, there exists a continuous function*

$$\omega(\varepsilon) \to 1 \quad (\varepsilon \to 0),$$

such that for $\lambda_\varepsilon := \lambda - \varepsilon\omega(\varepsilon)\|v^*\|$, there holds

$$\|(\mathscr{A}'(\hat{v}) - \lambda_\varepsilon\mathscr{I})^{-1}\| \geqslant \frac{1}{\varepsilon}. \tag{2.15}$$

Combining the two foregoing propositions, we obtain the following result. Under the assumption that $\mathrm{Re}\lambda_\varepsilon = \mathrm{Re}\lambda - \varepsilon\mathrm{Re}\,\omega(\varepsilon)\|v^*\| < 0$, there holds

$$\sup_{t \geqslant 0} \|S(t)\| \geqslant \frac{|\mathrm{Re}\lambda_\varepsilon|}{|\varepsilon|}. \tag{2.16}$$

Therefore, for $\varepsilon := \mathrm{Re}\lambda > 0$, we have

$$\mathrm{Re}\lambda_\varepsilon = \mathrm{Re}\lambda - \mathrm{Re}\lambda\,\mathrm{Re}\,\omega(\mathrm{Re}\lambda)\|v^*\| = \mathrm{Re}\lambda\left(1 - \mathrm{Re}\,\omega(\mathrm{Re}\lambda)\|v^*\|\right) < 0,$$

for $\|v^*\|$ sufficiently large. Consequently,

$$\sup_{t\geqslant 0}\|S(t)\| \geqslant \left|1 - \mathrm{Re}\,\omega(\mathrm{Re}\lambda)\|v^*\|\right|. \tag{2.17}$$

Hence, for small $\mathrm{Re}\lambda > 0$, a large value $\|v^*\| \gg 1$ indicates a large growth constant A and consequently possible nonlinear instability.

3. GALERKIN FINITE ELEMENT DISCRETIZATION

The finite element discretization of problems (2.1) and (2.5) uses decompositions $\mathbb{T}_h$ of $\bar{\Omega}$ into cells T (triangles, quadrilaterals, etc.). The local width of a cell $T \in \mathbb{T}_h$ is h_T, while $h := \max_{T\in\mathbb{T}_h} h_T$ denotes the global mesh size. For simplicity, we consider here only low-order tensor-product elements, that is piecewise d-linear trial and test functions for all unknowns (so-called 'Q_1/Q_1-Stokes element'). In order to ease local mesh refinement and coarsening, we allow 'hanging' nodes (see Figure 1), where the corresponding 'irregular' nodal values are eliminated from the system by linear interpolation of neighboring regular nodal values. The corresponding finite element subspaces are denoted by

$$L_h \subset L,\ \hat{\mathbf{H}}_h \subset \hat{\mathbf{H}},\ \mathbf{H}_h \subset \mathbf{H}, \qquad \hat{\mathbf{V}}_h := \hat{\mathbf{H}}_h \times L_h,\ \mathbf{V}_h := \mathbf{H}_h \times L_h,$$

and $\hat{v}_h^{\mathrm{in}} \in \hat{\mathbf{H}}_h$ is a suitable interpolation of the boundary function $\hat{v}^{\mathrm{in}}$. This construction is oriented by the situation of a polygonal domain Ω for which the boundary $\partial\Omega$ is exactly matched by the mesh domain $\Omega_h := \cup\{T \in \mathbb{T}_h\}$. In the case of a curved boundary some standard modifications are necessary, the description of which is omitted here. For more details on the terminology of finite element discretization, we refer to Ciarlet [10] or Girault & Raviart [15].

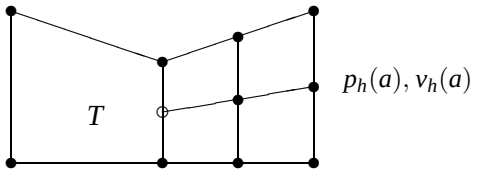

Figure 1. Quadrilateral mesh patch for the Q_1/Q_1-Stokes element with a 'hanging node'.

Since the finite element approximations $v_h \in \mathbf{H}_h$ of $v \in \mathbf{J}_1$ are usually not exactly solenoidal, we have to also consider the approximate pressures $p_h \in L_h$. Therefore, from now on we will consider approximating pairs $u_h = \{v_h, p_h\} \in \mathbf{V}_h$

to $u = \{v, p\} \in \mathbf{V}$. Correspondingly, the Navier-Stokes system is written in operator form like

$$\mathscr{A}\hat{u} := -\nu\Delta\hat{v} + \hat{v}\cdot\nabla\hat{v} + \nabla\hat{p} = f, \quad \nabla\cdot\hat{v} = 0 \quad \text{in } \Omega, \tag{3.1}$$

and the associated eigenvalue problem like

$$\mathscr{A}'(\hat{u})u := -\nu\Delta v + \hat{v}\cdot\nabla v + v\cdot\nabla\hat{v} + \nabla p = \lambda v, \quad \nabla\cdot v = 0 \quad \text{in } \Omega. \tag{3.2}$$

In order to obtain a stable discretization of (2.1) in these spaces with 'equal-order interpolation' of pressure and velocity, we use the least-squares technique proposed by Hughes, Franca & Balestra [21]. Following Hughes & Brooks [20], a similar approach is employed for stabilizing the convection term. We use the approximation

$$\mathscr{S}(\hat{u})\varphi := \hat{v}\cdot\nabla\varphi^v + \nabla\varphi^p, \quad \varphi = \{\varphi^v, \varphi^p\} \in \mathbf{V},$$

to the derivative $\mathscr{A}'(\hat{u})$ for defining the stabilized form

$$a_h(\hat{u};\varphi) := a(\hat{u};\varphi) + (\mathscr{A}(\hat{u}) - f, \mathscr{S}(\hat{u})\varphi)_h,$$

with the mesh-dependent inner product and norm

$$(\varphi,\psi)_h := \sum_{T\in\mathbb{T}_h} \delta_K\,(\varphi,\psi)_T, \quad \|\varphi\|_h = (\varphi,\varphi)_h^{1/2}.$$

With this notation the discrete eigenvalue problem (3.6) uses the sesquilinear form

$$a_h'(\hat{v}_h; u_h, \varphi_h) := a'(\hat{u}; u_h, \varphi) + (\mathscr{A}'(\hat{u})u - \lambda_h v, \mathscr{S}(\hat{u})\varphi)_h$$

which is not the derivative of the stabilized form $a_h(\cdot;\cdot)$, but rather a consistent stabilization of $a'(\hat{u};\cdot,\cdot)$. The stabilization parameters δ_T are chosen according to

$$\delta_T = \alpha\big(\nu h_T^{-2}, \beta\,|v_h|_{T;\infty} h_T^{-1}\big)^{-1}, \quad \delta := \max_{T\in\mathbb{T}_h}\delta_T, \tag{3.3}$$

with the heuristic values $\alpha = \frac{1}{12}$, $\beta = \frac{1}{6}$. Then, the discrete Navier-Stokes problem determines $\hat{u}_h := \{\hat{v}_h, \hat{p}_h\} \in V_h + \{\hat{v}_h^{\text{in}}, 0\}$ by

$$a_h(\hat{u}_h;\varphi_h) = F(\varphi_h) \quad \forall\varphi_h \in \mathbf{V}_h. \tag{3.4}$$

This discretization is fully consistent with (2.1) in the sense that the continuous solution $\hat{u}$ automatically satisfies (3.4). This implies 'Galerkin orthogonality' what in this case means

$$a_h(\hat{u};\varphi_h) - a_h(\hat{u}_h;\varphi_h) = 0, \quad \varphi_h \in \mathbf{V}_h. \tag{3.5}$$

The associated discrete primal and dual eigenvalue problems seek $u_h = \{v_h, p_h\}$ and $u_h^* = \{v_h^*, p_h^*\}$ in $\mathbf{V}\backslash\{0\}$ and $\lambda_h, \lambda_h^* \in \mathbb{C}$, such that

$$a_h'(\hat{u}_h; u_h, \varphi_h) = \lambda_h m(u_h, \varphi_h) \quad \forall \varphi_h \in \mathbf{V}_h, \tag{3.6}$$

$$a_h'(\hat{u}_h; \varphi_h, u_h^*) = \lambda_h^* m(\varphi_h, u_h^*) \quad \forall \varphi_h \in \mathbf{V}_h, \tag{3.7}$$

where $m(u_h, \varphi_h) := (v_h, \varphi_h^v)$ and $m(\varphi_h, u_h^*) := (\varphi_h^v, v_h^*)$. The eigenfunctions are usually normalized by $m(u_h, u_h) = m(u_h, u_h^*) = 1$, assuming nondegeneracy of the approximate eigenvalue λ_h. The question of possibly non-zero defect of this eigenvalue will be addressed below.

For this approximation, we can recall *a priori* error estimates from the literature. If the problem is H^2-regular, there holds an optimal-order error estimate for the approximation of the base solution (see Girault & Raviart [15] or Rannacher [31]),

$$\|\hat{v}_h - \hat{v}\| = \mathcal{O}(h^2), \tag{3.8}$$

and for a non-deficient eigenvalue (see Osborn [30]),

$$|\lambda_h - \lambda| = \mathcal{O}(h^2). \tag{3.9}$$

Further, for normalized discrete primal and dual eigenfunctions $\{u_h, u_h^*\} \in \mathbf{V}_h \times \mathbf{V}_h$, there exists an associated pair of eigenfunctions $\{u^h, u^{h*}\} \in \mathbf{V} \times \mathbf{V}$, such that

$$\|v_h - v^h\| + \|v_h^* - v^{h*}\| = \mathcal{O}(h^2). \tag{3.10}$$

Here, the superscript in v^h indicates that the continuous eigenfunction associated to v_h may vary with h. For a nondeficient eigenvalue λ with multiplicity m, there are exactly m approximating eigenvalues $\{\lambda_h^i\}_{i=1,\dots,m}$, counted according to their algebraic multiplicities, such that

$$\Big| \sum_{i=1}^{m} \lambda_h^i - \lambda \Big| = \mathcal{O}(h^2). \tag{3.11}$$

If λ has non-trivial defect, we have to construct approximations $u_h^{ij} \in \mathbf{V}_h$ to higher-order generalized eigenfunctions of λ. This construction will be described below.

In the above estimates the linearization is assumed to be exact, that is also in the approximate eigenvalue problem (3.6) the derivative of $a(\cdot;\cdot)$ is taken at the true solution $\hat{u}$. In the following *a posteriori* error analysis, we will treat the full discretization error incorporating also the linearization due to replacing $\hat{u}$ by $\hat{u}_h$. For simplifying the presentation, we will only consider the case of purely homogeneous Dirichlet boundary conditions, $\Gamma_{\text{rigid}} = \partial\Omega$, and will omit the terms related to the stabilization, that is, we will set $\delta = 0$.

4. AN ABSTRACT APPROACH TO A POSTERIORI ERROR CONTROL

From Becker & Rannacher [7], we recall an abstract framework for the *a posteriori* error analysis of Galerkin approximation of general nonlinear variational equations. Let $A(\cdot;\cdot)$ be a differentiable semilinear form and $F(\cdot)$ a linear functional defined on some real or complex function space $\mathscr{V}$. The derivatives of $A(\cdot;\cdot)$ at a point U in direction Φ are denoted by $A'(U;\Phi,\cdot)$, $A''(U;\Phi,\cdot,\cdot)$, and $A'''(U;\Phi,\cdot,\cdot,\cdot)$. We assume that the variational equation

$$A(U;\Phi) = F(\Phi) \quad \forall \Phi \in \mathscr{V}, \tag{4.1}$$

has a solution $U \in \mathscr{V}$ that is not required to be unique. The latter flexibility is essential for the application of the theory to be developed for eigenvalue problems. Suppose that the goal is to compute a certain physical quantity extracted from the solution U by applying a differentiable functional $J(\cdot)$, with derivatives at U denoted by $J'(U;\cdot)$, $J''(U;\cdot,\cdot)$, and $J'''(U;\cdot,\cdot,\cdot)$.

Problem (4.1) is approximated by a Galerkin method using finite dimensional subspaces $\mathscr{V}_h \subset \mathscr{V}$ parameterized by $h \in \mathbb{R}_+$. The associated discrete problems

$$A(U_h;\Phi_h) = F(\Phi_h) \quad \forall \Phi_h \in \mathscr{V}_h, \tag{4.2}$$

are also assumed to possess solutions $U_h \in \mathscr{V}_h$, again not necessarily uniquely determined. Then, $J(U_h)$ is taken as approximation to the target quantity $J(U)$. The aim is now to derive an *a posteriori* estimate for the error $J(U) - J(U_h)$.

Remark 4.1. In view of the possible nonuniqueness of the solutions U and U_h, the formulated goal of estimating the error quantity $J(U) - J(U_h)$ needs some explanation. The *a posteriori* error representation to be derived below does not explicitly require that the approximation U_h is close to U. However, since it contains a remainder term in which the difference $U - U_h$ occurs, the result is useful only under the assumption that the convergence $U_h \to U$, as $h \to 0$, is known by *a priori* arguments.

For deriving a representation for the error $J(U)-J(U_h)$, we employ the Euler-Lagrange approach typically used in optimal control theory. Computing $J(U)$ from the solution of (4.1) is equivalent to determining stationary points of the Lagrangian functional

$$\mathscr{L}(U;Z) := J(U) + F(Z) - A(U;Z),$$

with the dual variable $Z \in \mathscr{V}$. In this framework, we seek solutions $\{U,Z\} \in \mathscr{V}\times\mathscr{V}$ to the Euler-Lagrange system

$$A(U;\Phi) = F(\Phi) \qquad \forall \Phi \in \mathscr{V}, \tag{4.3}$$

$$A'(U;\Phi,Z) = J'(U;\Phi) \quad \forall \Phi \in \mathscr{V}. \tag{4.4}$$

We note that the first equation of this system is just the considered variational equation (4.1). The Galerkin approximation of system (4.3,4.4) in the subspace $\mathscr{V}_h \subset \mathscr{V}$ seeks pairs $\{U_h, Z_h\} \in \mathscr{V}_h \times \mathscr{V}_h$, satisfying

$$A(U_h;\Phi_h) = F(\Phi_h) \qquad \forall \Phi_h \in \mathscr{V}_h, \tag{4.5}$$

$$A'(U_h;\Phi_h,Z_h) = J'(U_h;\Phi_h) \quad \forall \Phi_h \in \mathscr{V}_h. \tag{4.6}$$

To the approximate solutions $U_h \in \mathscr{V}_h$ of (4.5) and $Z_h \in \mathscr{V}_h$ of (4.6), we associate the residuals

$$\rho(U_h;\Phi) := F(\Phi) - A(U_h;\Phi),$$
$$\rho^*(Z_h;\Phi) := J'(U_h;\Phi) - A'(U_h;\Phi,Z_h),$$

which are defined for $\Phi \in \mathscr{V}$. For $\Phi_h \in \mathscr{V}_h$, we have $\rho(U_h;\Phi_h) = \rho^*(Z_h;\Phi_h) = 0$, by definition. For this situation, we have the following fundamental result of Becker & Rannacher [7].

Proposition 4.1. *For the Galerkin approximation (*4.5, 4.6*) of the saddle-point problem (*4.3,4.4*), we have the a posteriori error representation*

$$J(U) - J(U_h) = \tfrac{1}{2}\rho(U_h;Z-\Psi_h) + \tfrac{1}{2}\rho^*(Z_h;U-\Phi_h) + R_h, \tag{4.7}$$

for arbitrary elements $\Psi_h, \Phi_h \in \mathscr{V}_h$. The remainder term R_h is given by

$$\begin{aligned} R_h := \tfrac{1}{2}\int_0^1 \big\{ & J'''(U_h+sE_h;E_h,E_h,E_h) - A'''(U_h+sE_h;E_h,E_h,E_h,Z_h+sE_h^*) \\ & - 3A''(U_h+sE_h;E_h,E_h,E_h^*)\big\}\, s(s-1)\,ds, \end{aligned} \tag{4.8}$$

where $E_h := U - U_h$ and $E_h^* := Z - Z_h$.

Proof. The proof is given in two steps. (i) First, we derive a general error representation for the Galerkin approximation of stationary points of functionals. Let $L(\cdot)$ be a differentiable functional on a (real or complex) function space X and $x \in X$ a stationary point, that is

$$L'(x;y) = 0 \quad \forall y \in X. \tag{4.9}$$

Further, let X_h be a finite dimensional subspace, indicated by a discretization parameter $h \in \mathbb{R}_+$, and let $x_h \in X_h$ be stationary points of $L(\cdot)$ on X_h, that is

$$L'(x_h;y_h) = 0 \quad \forall y_h \in X_h. \tag{4.10}$$

For the error $e_h := x - x_h$, we have the identity

$$L(x) - L(x_h) = \int_0^1 L'(x_h + se_h; e_h)\, ds.$$

Adding zero terms on the right-hand side results in

$$L(x)-L(x_h)=\tfrac{1}{2}L'(x_h;e_h)+\int_0^1 L'(x_h+se_h;e_h)\,ds-\tfrac{1}{2}L'(x_h;e_h)-\tfrac{1}{2}L'(x;e_h)\,.$$

We observe that the last two terms on the right are just the approximation of the second one by the trapezoidal rule. Recalling the corresponding remainder term

$$R_h=\tfrac{1}{2}\int_0^1 L'''(x_h+se_h;e_h,e_h,e_h)\,s(s-1)\,ds,$$

and noting that $L'(x_h;x_h-y_h)=0$, we obtain the error representation

$$L(x)-L(x_h)=\tfrac{1}{2}L'(x_h;x-y_h)+R_h,\quad y_h\in X_h. \tag{4.11}$$

(ii) Next, we give the proof of the representation (4.8). To this end, we embed problems (4.3,4.4) and (4.5,4.6) into the framework laid out in step (i). To this end, we define the spaces $X:=\mathscr{V}\times\mathscr{V}$ and $X_h:=\mathscr{V}_h\times\mathscr{V}_h$, such that the Lagrangian $\mathscr{L}(\cdot;\cdot)$ defines a functional $L(\cdot):=\mathscr{L}(\cdot;\cdot)$ on X. Correspondingly the solutions $x:=\{U,Z\}\in X$ of (4.3,4.4) and $x_h:=\{U_h,Z_h\}\in X_h$ of (4.5,4.6) are stationary points of $L(\cdot)$ on X and X_h, respectively. At these solutions, there holds

$$\begin{aligned}L(x)-L(x_h)&=\mathscr{L}(U;Z)-\mathscr{L}(U_h;Z_h)\\&=J(U)+F(Z)-A(U;Z)-J(U_h)-F(Z_h)+A(U_h;Z_h)\\&=J(U)-J(U_h).\end{aligned}$$

Therefore, from the general error representation (4.11), we obtain that

$$\begin{aligned}J(U)-J(U_h)&=\tfrac{1}{2}L'(x_h;x-y_h)+R_h\\&=\tfrac{1}{2}\rho(U_h,Z-\Psi_h)+\tfrac{1}{2}\rho^*(Z_h;U-\Phi_h)+R_h,\end{aligned}$$

with arbitrary $y_h=\{\Phi_h,\Psi_h\}\in\mathscr{V}_h\times\mathscr{V}_h$. Since $\mathscr{L}(\cdot;\cdot)$ is linear in its second argument, the third derivative of $L(\cdot)$ consists of only three terms,

$$\begin{aligned}L'''(x_h+s\hat{e}_h;\hat{e}_h,\hat{e}_h,\hat{e}_h)=J'''(U_h+sE_h;E_h,E_h,E_h)-3A''(U_h+sE_h;E_h,E_h,E_h^*)\\-A'''(U_h+sE_h;E_h,E_h,E_h,Z_h+sE_h^*).\end{aligned}$$

Therefore, the remainder term R_h has the asserted form (4.8). □

5. APPLICATION TO THE NAVIER-STOKES EQUATIONS

We recall the situation described at the end of Section 3 where m distinct simple eigenvalues λ_h^i $(i=1,\dots,m)$ approximate a limit eigenvalue λ. The corresponding pairs of primal and dual eigenfunctions u_h^i, u_h^{i*} are normalized by $m(u_h^i,u_h^i)=m(u_h^i,u_h^{i*})=1$. We note that the possible blow-up

$$m(u_h^{i*},u_h^{i*})\to\infty\quad(h\to 0), \tag{5.1}$$

indicates defect $\alpha > 0$ of the limiting eigenvalue λ. If this is observed, then the *a posteriori* error analysis has to be accordingly organized. The analysis of Section 2.1 has shown that also

$$\sup_{h\in\mathbb{R}_+} m(u_h^{i*}, u_h^{i*}) \gg 1 \tag{5.2}$$

can be taken as indication that the limiting eigenvalue λ, though $0 < \mathrm{Re}\lambda \ll 1$, may cause a large amplification constant A and consequently nonlinear instability.

Our error analysis for the approximation of the Navier-Stokes equations and the associated primal and dual eigenvalue problems relies on their interpretation as nonlinear equations and their embedding into the framework laid out in the previous section. In order to clearly present the ideas underlying our approach, we first consider the case that there is a single approximating eigenvalue with multiplicity $m = 1$, such that also the limit eigenvalue λ must be simple and non-degenerate. This easily generalizes to the case of a geometrically multiple but non-degenerate eigenvalue. The critical case that the limiting eigenvalue λ is degenerate requires a substantial modification of the argument. For the details, we refer to Heuveline & Rannacher [18].

Suppose that, for sufficiently small $h \in \mathbb{R}_+$, the distinct (simple) eigenvalues λ_h^i $(i = 1,\dots,m)$ are observed to converge to some limit eigenvalue λ, for $h \to 0$. The corresponding primal and dual eigenfunctions u_h^i and u_h^{i*} are normalized by $m(u_h^i, u_h^i) = m(u_h^i, u_h^{i*}) = 1$. To these pairs, we can associate corresponding primal and dual continuous eigenfunctions u^i, u^{i*} of the limit eigenvalue λ, such that (3.10) holds. Suppose that we observe

$$\sup_{h\in\mathbb{R}_+} m(u_h^{i*}, u_h^{i*}) \leqslant K \quad (i = 1,\dots,m), \tag{5.3}$$

with some constant K of moderate size, which indicates that λ is non-deficient. In this case, we need to consider only a single approximating eigenvalue $\lambda_h \in \{\lambda_h^1,\dots,\lambda_h^m\}$ and a corresponding pair of approximating primal and dual eigenvectors u_h, u_h^*, in order to achieve the full order of convergence.

For embedding this situation into the framework laid out in the previous section, we introduce the spaces $\mathscr{V} := \mathbf{V}\times\mathbf{V}\times\mathbb{C}$ and $\mathscr{V}_h := \mathbf{V}_h\times\mathbf{V}_h\times\mathbb{C}$ and denote their elements by $U := \{\hat{u}, u, \lambda\}$ and $U_h := \{\hat{u}_h, u_h, \lambda_h\}$, respectively. Further, for $\Phi = \{\hat{\varphi}, \varphi, \mu\} \in \mathscr{V}$, we introduce a semilinear form $A(\cdot;\cdot)$ by

$$A(U;\Phi) := f(\hat{\varphi}) - a(\hat{u};\hat{\varphi}) - a'(\hat{u};u,\varphi) + \lambda\, m(u,\varphi) + \overline{\mu}\big\{m(u,u) - 1\big\}.$$

With this notation, the system of equations (2.1), and (2.5) can be written in compact form as

$$A(U;\Phi) = 0 \quad \forall\Phi \in \mathscr{V}. \tag{5.4}$$

Correspondingly, the system of discrete equations (3.4) and (3.6) reads

$$A(U_h;\Phi_h) = 0 \quad \forall\Phi_h \in \mathscr{V}_h. \tag{5.5}$$

Clearly, the continuous as well as the discrete problem (5.4) and (5.5) have multiple solutions. In order to make the following analysis meaningful, we have to assume that there are particular solutions denoted by U and U_h, such that the error $E_h := U - U_h$ becomes small as $h \to 0$. This assumption is justified by the results of the *a priori* error analysis stated in Section 2.

Now, letting $J(\cdot)$ be a differentiable functional on $\mathscr{V}$, Proposition 4.1 gives us an identity for the error $J(U) - J(U_h)$ in terms of the corresponding primal and dual residuals $\rho(U_h;\cdot)$ and $\rho^*(Z_h;\cdot)$, respectively. This general result will be used for deriving a representation for the eigenvalue error $\lambda - \lambda_h$. To this end, we define the special functional

$$J(\Phi) := \mu\, m(\varphi,\varphi), \quad \Phi = \{\hat{\varphi},\varphi,\mu\} \in \mathscr{V}.$$

This is motivated by the fact that $J(U) = \lambda$, since $m(u,u) = 1$. The corresponding dual solutions $Z = \{\hat{z},z,\pi\} \in \mathscr{V}$ and $Z_h = \{\hat{z}_h,z_h,\pi_h\} \in \mathscr{V}_h$ are determined by the equation

$$A'(U;\Phi,Z) = J'(U;\Phi) \quad \forall \Phi \in \mathscr{V}, \tag{5.6}$$

and by its discrete analogue

$$A'(U_h;\Phi_h,z_h) = J'(U_h;\Phi_h) \quad \forall \Phi_h \in \mathscr{V}_h, \tag{5.7}$$

respectively. For our choice of $J(\cdot)$, we have

$$J'(U;\Psi) = \chi\, m(u,u) + 2\lambda\, \mathrm{Re}\, m(u,\psi).$$

Further, the derivative of $A(\cdot;\cdot)$ has the explicit form

$$\begin{aligned} A'(U;\Psi,Z) = &-a'(\hat{u};\hat{\psi},\hat{z}) - a''(\hat{u};\hat{\psi},u,z) \\ &- a'(\hat{u};\psi,z) + \lambda\, m(\psi,z) + \chi\, m(u,z) + 2\,\bar{\pi}\,\mathrm{Re}\, m(u,\psi), \end{aligned}$$

for arguments $Z = \{\hat{z},z,\pi\}$ and $\Psi = \{\hat{\psi},\psi,\chi\}$. With this notation, and observing that $m(u,u) = 1$, the dual problem (5.6) is equivalent to the system

$$a'(\hat{u};\psi,\hat{z}) = -a''(\hat{u};\psi,u,z) \quad \forall \psi \in \mathbf{V}, \tag{5.8}$$

$$-a'(\hat{u};\psi,z) + \lambda\, m(\psi,z) = 2(\lambda - \bar{\pi})\,\mathrm{Re}\{m(u,\psi)\} \quad \forall \psi \in \mathbf{V}, \tag{5.9}$$

$$\chi\, m(u,z) = \chi\, m(u,u) \quad \forall \chi \in \mathbb{C}. \tag{5.10}$$

Equation (5.9) reduces to the dual eigenvalue problem for $u^* := z$,

$$a'(\hat{u};\psi,u^*) = \lambda\, m(\psi,u^*) \quad \forall \psi \in \mathbf{V}, \tag{5.11}$$

while (5.10) yields the normalization condition $m(u,u^*) = 1$. Having identified $z = u^*$, the first equation (5.8) determines a dual solution $\hat{u}^* := \hat{z} \in \mathbf{V}$ by

$$a'(\hat{u};\psi,\hat{u}^*) = -a''(\hat{u};\psi,u,u^*) \quad \forall \psi \in \mathbf{V}. \tag{5.12}$$

Since

$$a''(\hat{u};\psi,u,u^*) = (\psi^v\cdot\nabla v,v^*)+(v\cdot\nabla\psi^v,v^*),$$

(5.12) can be written in the form

$$a'(\hat{u};\psi,\hat{u}^*) = (\psi^v,g(v,v^*)) \quad \forall\psi\in\mathbf{V}, \tag{5.13}$$

where $g(v,v^*) := v^*\cdot(\nabla v)^T - v\cdot\nabla v^* - (\nabla\cdot v)v^*$. The corresponding discrete dual problems are

$$a'(\hat{u}_h;\psi_h,u_h^*) = \lambda_h m(\psi_h,u_h^*) \quad \forall\psi_h\in\mathbf{V}_h, \tag{5.14}$$

with the normalization condition $m(u_h,u_h^*)=1$, and

$$a'(\hat{u}_h;\psi_h,\hat{u}_h^*) = (\psi_h^v,g(v_h,v_h^*)) \quad \forall\psi_h\in\mathbf{V}_h. \tag{5.15}$$

We introduce the primal residuals

$$\begin{aligned}\rho(\hat{u}_h;\psi) &:= f(\psi)-a(\hat{u}_h;\psi),\\ \rho(u_h,\lambda_h;\psi) &:= \lambda_h m(u_h,\psi)-a'(\hat{u}_h;u_h,\psi),\end{aligned}$$

and the associated dual residuals

$$\begin{aligned}\rho^*(\hat{u}_h^*;\psi) &:= -(\psi^v,g(v_h,v_h^*))-a'(\hat{u}_h;\psi,\hat{u}_h^*),\\ \rho^*(u_h^*,\lambda_h^*;\psi) &:= \lambda_h m(\psi,u_h^*)-a'(\hat{u}_h;\psi,u_h^*),\end{aligned}$$

defined for $\psi\in V$. After this preparation, we give a general representation for the eigenvalue error.

Proposition 5.1. *Let $\hat{u}$ be a solution of (*2.1*) and $\hat{u}_h$ its approximation determined by (*3.4*). Accordingly, let $\{u_h,\lambda_h\}$ and $\{u_h^*,\lambda_h^*\}$ be solutions of the primal discrete eigenvalue problem (*3.6*) and the associated dual problem (*3.7*), respectively, which are assumed to be simple. Then, for solutions u, u^* of the continuous eigenvalue problems (*2.5*), (*2.6*) and solutions $\hat{u}^*$ and $\hat{u}_h^*$ of the dual problems (*5.13*), (*5.15*), there holds the error identity*

$$\begin{aligned}\lambda-\lambda_h = &\tfrac{1}{2}\{\rho(\hat{u}_h;\hat{u}^*-\hat{\psi}_h)+\rho^*(\hat{u}_h^*;\hat{u}-\hat{\varphi}_h)\}\\ &+\tfrac{1}{2}\{\rho(u_h,\lambda_h;u^*-\psi_h)+\rho^*(u_h^*,\lambda_h^*;u-\varphi_h)\}+R_h,\end{aligned} \tag{5.16}$$

for arbitrary $\hat{\psi}_h$, ψ_h, $\hat{\varphi}_h$, $\varphi_h\in V_h$. The remainder R_h is given by

$$R_h = \tfrac{1}{2}e^{\lambda}(e^v,e^{v*})-\tfrac{1}{2}(\hat{e}^v\cdot\nabla\hat{e}^v,\hat{e}^{v*})-\tfrac{1}{2}(\hat{e}^v\cdot\nabla e^v,e^{v*})-\tfrac{1}{2}(e^v\cdot\nabla\hat{e}^v,e^{v*}),$$

where $e^{\lambda}:=\lambda-\lambda_h$, $\hat{e}^v:=\hat{v}-\hat{v}_h$, $\hat{e}^{v*}:=\hat{v}^*-\hat{v}_h^*$, $e^v:=v-v_h$, and $e^{v*}:=v^*-v_h^*$.

Proof. From Proposition 4.1, we recall the general identity

$$J(U)-J(U_h)=\tfrac{1}{2}\rho(U_h;Z-\Psi_h)+\tfrac{1}{2}\rho^*(Z_h;U-\Phi_h)+R_h,$$

with arbitrary $\Psi_h,\Phi_h\in\mathcal{Y}_h$, and the remainder term

$$\begin{aligned}R_h:=\tfrac{1}{2}\int_0^1\Big\{&J'''(U_h+sE;E,E,E)-3A''(U_h+sE;E,E,E^*)\\&-A'''(U_h+sE;E,E,E,Z_h+sE^*)\Big\}s(s-1)\,ds,\end{aligned}$$

where $E:=U-U_h$ and $E^*:=Z-Z_h$. At first, we note that, in view of the normalization conditions $m(u,u)=1$ and $m(u_h,u_h)=1$,

$$J(U)-J(U_h)=\lambda-\lambda_h.$$

The residuals $\rho(U_h;\Psi)$ and $\rho^*(Z_h;\Phi)$ have the explicit form

$$\rho(U_h;\Psi)=-A(U_h;\Psi)=a(\hat u_h;\hat\psi)-f(\hat\psi)+a'(\hat u_h;u_h,\psi)-\lambda_h m(u_h,\psi),$$

and

$$\begin{aligned}\rho^*(Z_h;\Phi)&=J'(U_h;\Phi)-A'(U_h;\Phi,Z_h)\\&=a''(\hat u_h;\hat\varphi,u_h,u_h^*)+a'(\hat u_h;\hat\varphi,\hat u_h^*)+a'(\hat u_h;\varphi,u_h^*)-\lambda_h m(\varphi,u_h^*),\end{aligned}$$

for $\Psi=\{\hat\psi,\psi,\chi\}$ and $\Phi=\{\hat\varphi,\varphi,\mu\}$. Finally, we have to evaluate the remainder term. By a simple calculation, we have

$$J'''(U_h+sE;E,E,E)=6(\lambda-\lambda_h)\|v-v_h\|^2,$$

and, since the semilinear form $A(U;\cdot)$ is quadratic in U,

$$A'''(U_h+sE;E,E,E,Z_h+sE^*)=0.$$

Further, since $a(\hat u;\cdot)$ is quadratic in $\hat u$,

$$\begin{aligned}A''(U;\Phi,\Psi,Z)=&-a''(\hat u;\hat\varphi,\hat\psi,\hat z)-a''(\hat u;\hat\varphi,\psi,z)-a''(\hat u;\varphi,\hat\psi,z)\\&+\mu\,m(\psi,z)+\chi\,m(\varphi,z)+2\,\overline{\pi}\,\mathrm{Re}\{m(\varphi,\psi)\},\end{aligned}$$

and consequently,

$$\begin{aligned}-3A''(U_h+sE;E,E,E^*)=&\,6(\hat e^v\cdot\nabla\hat e^v,\hat e^{v*})+6(\hat e^v\cdot\nabla e^v,e^{v*})+6(e^v\cdot\nabla\hat e^v,e^{v*})\\&-6(\lambda-\lambda_h)\,(v-v_h,v^*-v_h^*)-6(\lambda-\lambda_h)\|v-v_h\|^2.\end{aligned}$$

Collecting the above results, we obtain

$$\begin{aligned}R_h&=\tfrac{1}{2}\int_0^1\big\{J'''(U_h+sE;E,E,E)-3A''(U_h+sE;E,E,E^*)\big\}\,s(s-1)\,ds\\&=\tfrac{1}{2}e^\lambda\,(e^v,e^{v*})-\tfrac{1}{2}\big\{(\hat e^v\cdot\nabla\hat e^v,\hat e^{v*})+(\hat e^v\cdot\nabla e^v,e^{v*})+(e^v\cdot\nabla\hat e^v,e^{v*})\big\}.\end{aligned}$$

This completes the proof of (5.16). □

6. PRACTICAL ERROR ESTIMATION AND MESH ADAPTIVITY

In the following, we want to convert the general *a posteriori* error representation (5.16) into an *a posteriori* error estimate which can be used in practice. On this basis a strategy for automatic mesh refinement is developed. As described above, first, we have to detect whether the limit eigenvalue λ may degenerate. To this end, we construct the dual eigenfunctions $u_h^{i*} \in \mathbf{V}_h$ corresponding to the approximate eigenvalues λ_h^i $(i = 1,\dots,m)$ and check whether the blow-up (5.1) or (5.2) is observed for any index i. According to the observed behavior, we can use the *a posteriori* error representations of Proposition 5.1 or its refinement for the case of degenerate limit eigenvalue λ. The latter result is rather complicated and not further considered here. For the details, we refer to Heuveline & Rannacher [18]. In the following, we will only consider the case of Proposition 5.1 in more detail.

6.1. Evaluation of error representation

According to our consistency assumption, that is the convergence $\hat{u}_h \to \hat{u}$, $\hat{u}_h^* \to \hat{u}^*$, $u_h \to u$, and $u_h^* \to u^*$, as $h \to 0$, the higher-order remainder term R_h in the error representation (5.16) is supposed to be small and is therefore neglected. Let $i_h : \mathbf{H}^2(\Omega) \to \hat{\mathbf{H}}_h$ and $j_h : H^2(\Omega) \to L_h$ denote the generic operators of nodal interpolation in the finite element spaces $\hat{\mathbf{H}}_h$ and L_h, respectively. The evaluation of the residual terms is described for the term

$$\rho(\hat{u}_h; \hat{u}^* - \hat{\psi}_h) = F(\hat{u}^* - \hat{\psi}_h) - a(\hat{u}_h; \hat{u}^* - \hat{\psi}_h),$$

as a prototype. Splitting the integrals over Ω into a sum of contributions from each cell T, and integrating cell-wise by parts, we obtain for $\hat{w}^* := \{\hat{v}^* - i_h\hat{v}^*, \hat{p}^* - j_h\hat{p}^*\}$:

$$\begin{aligned}\rho(\hat{u}_h; \hat{w}) &= \sum_{T\in\mathbb{T}_h} \big\{ (f - \mathscr{A}(\hat{u}_h), \hat{w}^{v*})_T - (\nu\partial_n v_h, \hat{w}^{v*})_{\partial T} - (\hat{w}^{p*}, \nabla\cdot\hat{v}_h)_T \big\} \\ &= \sum_{T\in\mathbb{T}_h} \big\{ (f - \mathscr{A}(\hat{u}_h), \hat{w}^{v*})_T - \tfrac{1}{2}(\nu[\partial_n v_h], \hat{w}^{v*})_{\partial T} - (\hat{w}^{p*}, \nabla\cdot\hat{v}_h)_T \big\},\end{aligned}$$

where $[\partial_n v_h]_\Gamma := \partial_n v_{h|\Gamma} + \partial_{n'} v_{h|\Gamma}$ denotes the jump of $\partial_n v_h$ across the common edge of two cells, $\Gamma = T \cap T'$. For cells with an edge Γ on the boundary $\partial\Omega$, we use the convention that $[\partial_n v_h]_{|\Gamma} := 0$, if $\Gamma \subset \Gamma_{\text{rigid}} \cup \Gamma_{\text{in}}$, and $[\partial_n v_h]_{|\Gamma} := \partial_n v_h$, if $\Gamma \subset \Gamma_{\text{out}}$.

While the cell-residual quantities $f - \mathscr{A}(\hat{u}_h)$, $\nu[\partial_n v_h]$, and $\nabla\cdot\hat{v}_h$ can be computed exactly, the functions w^v and w^p depend on the unknown dual solution $\hat{u}^*$ and have to be approximated. However, this approximation does not need to be of very high accuracy since these terms only play the role of 'weights' for the residuals in the error representation. Here, we consider 'patchwise higher-order interpolation' as one of the simplest and cheapest options. Let $\hat{u}_h^*$ be the approximation computed on the current mesh. On square blocks of 2^d neighboring cells the 3^d nodal values of $\hat{v}_h$ and $\hat{p}_h$ are used to define d-quadratic interpolations $i_{2h}^{(2)}\hat{v}_h^*$ and $j_{2h}^{(2)}\hat{p}_h^*$,

respectively. These are then used in the approximation

$$\hat{v}^* - i_h\hat{v}^* \approx i_{2h}^{(2)}\hat{v}_h^* - \hat{v}_h^*, \qquad \hat{p}^* - j_h\hat{p}_h^* \approx j_{2h}^{(2)}\hat{p}_h^* - \hat{p}_h^*.$$

In many applications this approximation has proved to be sufficiently accurate. For a detailed discussion of this crucial aspect of using duality-based a posteriori error representations of the type (5.16), we refer to Bangerth & Rannacher [2]. The other residual terms $\rho^*(\hat{u}_h^*;\hat{u}-\hat{\varphi}_h)$, $\rho(u_h^i,\lambda_h^i;u^{i*}-\psi_h^i)$, and $\rho^*(u_h^{i*},\lambda_h^{i*};u^i-\varphi_h^i)$ can be treated analogously. This leads us to the following result.

Proposition 6.1. *With the notation of Proposition* 5.1*, we have the a posteriori error estimate:*

$$|\lambda-\lambda_h| \approx \eta(\hat{u}_h,\hat{u}_h^*,u_h,u_h^*,\lambda_h) := \Big|\sum_{T\in\mathbb{T}_h}\{\hat{\eta}_T+\eta_T^{\lambda}\}\Big|, \tag{6.1}$$

with the cell-error indicators

$$\begin{aligned}
\hat{\eta}_T &:= (f-\mathscr{A}(\hat{u}_h),\hat{w}^{v*})_T - \tfrac{1}{2}(\nu[\partial_n\hat{v}_h],\hat{w}^{v*})_{\partial T} - (\hat{w}^{p*},\nabla\cdot\hat{v}_h)_T\\
&\quad + (\hat{w}^{v},g(v_h,v_h^*)+\mathscr{A}'(\hat{u}_h)\hat{u}_h^*)_T - \tfrac{1}{2}(\hat{w}^{v},\nu[\partial_n\hat{v}_h^*])_{\partial T} - (\nabla\cdot\hat{v}_h^*,\hat{w}^{p})_T,\\
\eta_T^{\lambda} &:= (\lambda_h v_h-\mathscr{A}'(\hat{u}_h)u_h,w^{v*})_T - \tfrac{1}{2}(\nu[\partial_n v_h],w^{v*})_{\partial T} - (w^{p*},\nabla\cdot v_h)_T\\
&\quad + (w^{v},\lambda_h v_h^*-\mathscr{A}'(\hat{u}_h)u_h^*)_T - \tfrac{1}{2}(w^{v},\nu[\partial_n v_h^*])_{\partial T} - (\nabla\cdot v_h^*,w^{p})_T,
\end{aligned}$$

where $\hat{w}^{v*} := i_{2h}^{(2)}\hat{v}_h^* - \hat{v}_h^*$, $\hat{w}^{p*} := j_{2h}^{(2)}\hat{p}_h^* - \hat{p}_h^*$, $\hat{w}^{v} := \hat{w}^{v} := i_{2h}^{(2)}\hat{v}_h - \hat{v}_h$, $\hat{w}^{p} := j_{2h}^{(2)}\hat{p}_h - \hat{p}_h$, $w^{v*} := i_{2h}^{(2)}v_h^* - v_h^*$, $w^{p*} := j_{2h}^{(2)}p_h^* - p_h^*$.

Remark 6.1. An a posteriori error estimate analogous to that of Proposition 6.1 can also be derived for the case of a deficient eigenvalue λ. For the details see Heuveline & Rannacher [18].

Remark 6.2. Using the Hölder inequality together with the triangle inequality the a posteriori error estimator $\eta(\hat{u}_h,\hat{u}_h^*,u_h,u_h^*,\lambda_h)$ can be bounded in terms of the sums over products of the form $\hat{\rho}_T^{v}\,\hat{\omega}_T^{v*}$ with the cell-residuals and weights

$$\begin{aligned}
\hat{\rho}_T^{v} &:= \|f-\mathscr{A}(\hat{u}_h)\|_T + \frac{1}{2}h_T^{-1/2}\|\nu[\partial_n\hat{v}_h]\|_{\partial T},\\
\hat{\omega}_T^{v*} &:= \max\big\{\|\hat{v}^*-i_h\hat{v}^*\|_T, \tfrac{1}{2}h_T^{1/2}\|\hat{v}^*-i_h\hat{v}^*\|_{\partial T}\big\}.
\end{aligned}$$

However, this step may lead to significant over-estimation since it looses possible error cancellation between the various terms on each cell $T\in\mathbb{T}_h$. This estimation can be carried even further using the estimates for the nodal interpolation

$$\|v-i_h v\|_T + h_T^{1/2}\|v-i_h v\|_{\partial T} \leqslant c_i h_T^2\|\nabla^2 v\|_T, \quad v\in\mathbf{H}^2(T),$$

and the a priori bounds

$$\|\nabla^2 v\| \leqslant c_s \|f\|.$$

The latter estimate requires the given problem to be H^2-regular by which the case of non-convex polygonal or polyhedral domains are excluded. The result is an estimate for the error estimator $\eta(\hat{u}_h, \hat{u}_h^*, u_h, u_h^*, \lambda_h)$ by a sum over quantities of the form $h_T^2(\hat{\rho}_T^v)^2$, in which the unknown local weights $\hat{\omega}_T^{v*}$ are replaced by (as well unknown) global *stability constants*. In general, this estimate represents a rather crude overestimation of the actual eigenvalue error, particularly in the case of non-symmetric problems such as in the Navier-Stokes equations, and is not suitable for steering efficient mesh adaptation (see Heuveline & Rannacher [17]).

6.2. Refinement strategies

We briefly discuss our strategy for automatic mesh adaptation based on the approximate *a posteriori* error estimate (6.1) derived above. Let TOL be a given error tolerance and $N_{\max}$ the maximum number of mesh cells that can be used.

In the case $|\eta_T^\lambda| \ll |\hat{\eta}_T|$ the total error contribution by the cell T is dominated by its component due to the base flow approximation. This would suggest the use of different meshes in computing the base flow and solving the eigenvalue problem in order to decrease costs. The a posteriori error estimate (6.1) remains valid for such a hybrid discretization. Here, we will not further pursue this idea. Then, the mesh adaptation strategy aims to equilibrate the combined indicators

$$\eta_T := |\hat{\eta}_T| + |\eta_T^\lambda|$$

by locally refining or coarsening the mesh.

In the following, we consider quadrilateral or hexahedral meshes of a two- or three-dimensional polygonal or polyhedral domain Ω. Grid refinement is realized by edge-bisection, that is by cutting a cell T on mesh level l into 2^d regular cells on mesh level $l+1$. Correspondingly, mesh coarsening is realized by combining 2^d cells on mesh level l to one cell on mesh level $l-1$. This process may create cells with 'hanging nodes' such that the mesh is not compatible. The resulting nonconformity of the trial functions is avoided by eliminating the unknowns corresponding to any irregular node by linear interpolation of the values at the neighboring regular nodes. Then, the resulting finite element space is again 'conforming', that is $\mathbf{V}_h \subset \mathbf{V}$, for details see Heuveline/Schieweck [19]).

Starting from a coarse mesh $\mathbb{T}_0 := \mathbb{T}_{h_0}$ with mesh size distribution h_0, let after l refinement cycles the mesh-level $\mathbb{T}_l := \mathbb{T}_{h_l}$ with space $\mathbf{V}_l := \mathbf{V}_{h_l}$ be reached. Let $N_l \approx \dim(\mathbf{V}_l)$ be the number of cells of the mesh $\mathbb{T}_l$. On this mesh the approximate solution $\{\hat{u}_l, \hat{u}_l^*, u_l, u_l^*, \lambda_l\}$ is computed and the associated cell-error indicators $\hat{\eta}_T$ and η_T^λ are evaluated.

Stopping criterion: If the criterion

$$\Big|\sum_{T\in\mathbb{T}_h} \{\hat{\eta}_T + \eta_T^\lambda\}\Big| \leqslant \frac{1}{2}\mathrm{TOL}, \qquad \Big|\sum_{\mathrm{T}\in\mathbb{T}_\mathrm{h}} \hat{\eta}_\mathrm{T}\Big| \leqslant \Big|\sum_{\mathrm{T}\in\mathbb{T}_\mathrm{h}} \eta_\mathrm{T}^\lambda\Big| \tag{6.2}$$

is satisfied on the mesh $\mathbb{T}_l$, then the refinement process is stopped and λ_l is accepted as approximation to λ. Otherwise, the next refinement cycle is started.

Adaptation step: The transition from mesh $\mathbb{T}_l$ to the next mesh $\mathbb{T}_{l+1}$ follows the so-called 'fixed rate' strategy. Here, in each refinement cycle, the goal is to increase the number of mesh cells N_l by a fixed rate or to reduce the error estimator $\eta_l^{\lambda}(\hat{u}_l,\hat{u}_l^*,u_l,u_l^*,\lambda_l)$ by a fixed rate. First, the cells $T\in\mathbb{T}_l$ are ordered according to the size of the indicator values η_T,

$$\{T_i, i=1,...,N_l\}: \quad \eta_{T,1}\geqslant ...\geqslant \eta_{T,i}\geqslant \eta_{T,i+1}\geqslant ...\geqslant \eta_{T,N_l}.$$

For prescribed rates $X\%$ and $Y\%$ the cells are grouped according to

$$\#\{T_i, i=1,...,N_*\}\approx \frac{X}{100}N_l, \qquad \#\{T_{N_l-i+1}, i=1,...,N^*\}\approx \frac{Y}{100}N_l.$$

Then, the cells $T_1,\ldots,T_{N_*}$ are refined and the cells $T_{N_l-N^*+1},\ldots,N_l$ are coarsened. By this strategy the number of mesh cells changes with a prescribed rate which is advantageous in using a multigrid solver. In the test calculations discussed below, we have used $X=20$ and $Y=0$, i.e., only mesh refinement is performed.

7. NUMERICAL EXAMPLES

We present some results of calculations that are intended to illustrate the theoretical results derived in the preceding sections.

7.1. Test case 1: Diffusion-transport eigenvalue problem

For our computational tests, we consider the model problem

$$-\Delta v + b\cdot\nabla v = \lambda v \quad \text{in } \Omega, \quad v_{|\partial\Omega}=0, \tag{7.1}$$

with $b=(0,3)^T$, defined on the rectangular domain $\Omega=(-1,1)\times(-1,3)$ with a slit with tip at $(0,0)$. In the presence of a reentrant corner with angle $\omega=2\pi$, the solution contains a 'corner singularity', that is it can be written in the form $v=Ar^{1/2}\sin(\vartheta/2)+\tilde{v}$, with $\tilde{v}\in H^2(\Omega)$ and $\{r,\vartheta\}$ being polar coordinates. Further, for transport into y-direction a boundary layer occurs at the upper boundary $\{y=3\}$. Note that in this case, the corresponding dual eigenfunction v^* has a boundary layer at the lower boundary $\{y=-1\}$. Figure 2 contains plots of the (normalized) primal and dual eigenfunction corresponding to λ. We will compare the performance of the following three different error estimators for controlling the mesh construction in computing the eigenvalue:

$$\eta_\lambda^{(1)} := \sum_{T\in\mathbb{T}_h} h_T^2 \{\rho_T^2 + \rho_T^{*2}\}, \tag{7.2}$$

$$\eta_\lambda^{\text{red}} := \sum_{T\in\mathbb{T}_h} h_T^2 \rho_T^2, \tag{7.3}$$

$$\eta_\lambda^{\text{weight}} := \sum_{T\in\mathbb{T}_h} h_T^2 \{\rho_T \tilde{\omega}_T^* + \rho_T^* \tilde{\omega}_T\}. \tag{7.4}$$

The 'weighted' error estimator $\eta_\lambda^{\text{weight}}$ is derived from the eigenvalue error representation (6.1), while the error estimator $\eta_\lambda^{(1)}$ and its reduced version $\eta_\lambda^{\text{red}}$ are obtained by following the procedure described in Remark 6.2.

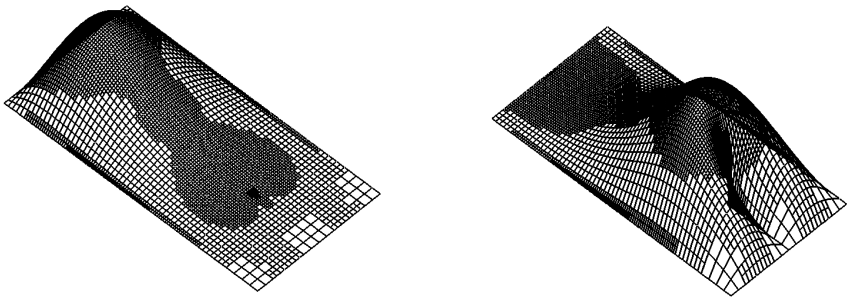

Figure 2. Primal (left) and dual (right) eigenfunctions for the non-symmetric case.

The 'best' meshes obtained by the error estimators $\eta_\lambda^{(1)}$, $\eta_\lambda^{\text{red}}$ and $\eta_\lambda^{\text{weight}}$ are shown in Figure 3. The 'fixed-rate strategy' with $X = 20\%$ and $Y = 0\%$ has been used in the refinement process. We see that the meshes are quite different, though all three indicators are based on more or less rigorous grounds. In Figure 4, we compare the efficiency of the error estimators against global (uniform) refinement. We see that the reduced estimator $\eta_\lambda^{\text{red}}$ shows significantly weaker performance than the other estimators what is due to not resolving the boundary layer of the dual eigenfunction v^*. As predicted, the weighted estimator $\eta_\lambda^{\text{weight}}$ is superior over the estimators $\eta_\lambda^{(1)}$. We also see that, though reflecting the correct asymptotic behavior for $N \to \infty$ ('reliability'), all three estimators crudely overestimate the true error ('efficiency'). This is due to the many steps of applying Hölder and triangle inequality in their derivation which lose sharpness. A more reliable estimate of the true eigenvalue error may be obtained by monitoring the convergence behavior of the approximate eigenvalues on the generated sequence of adapted meshes.

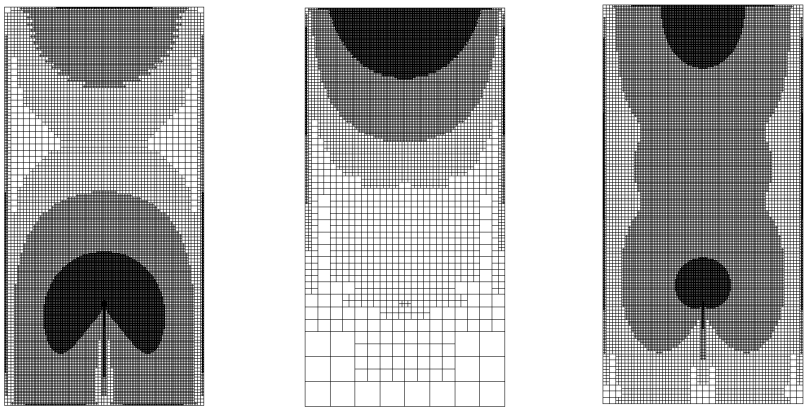

Figure 3. Adapted meshes with about $10,000$ cells constructed by the error estimator $\eta_\lambda^{(1)}$ (left), by its reduced version $\eta_\lambda^{\mathrm{red}}$ (middle) and by the 'weighted' estimator $\eta_\lambda^{\mathrm{weight}}$ (right).

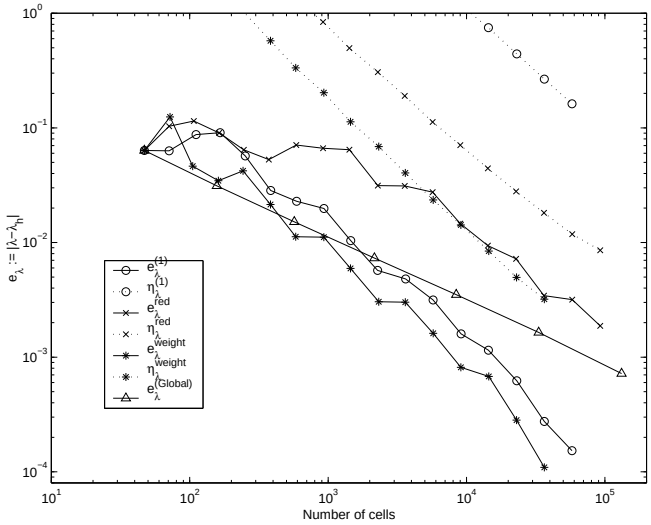

Figure 4. Mesh-efficiencies of the error estimators $\eta_\lambda^{(1)}$ (symbol 'O'), $\eta_\lambda^{\mathrm{red}}$ (symbol '$\times$'), $\eta_\lambda^{\mathrm{weight}}$ (symbol '$*$'), compared to uniform refinement (symbol '$\triangle$') for the non-symmetric case. The dashed curves indicate the values of the different error estimators.

7.2. Test case 2: Burgers equation

The next test case is the two-dimensional Burgers equation

$$-\nu\Delta\hat{v}+\hat{v}\cdot\nabla\hat{v}=0 \quad \text{in } \Omega, \tag{7.5}$$

on the rectangular domain $\Omega := (0,10)\times(0,1)\subset\mathbb{R}^2$, with Dirichlet boundary conditions along $\{x_1=0\}$, $\{x_2=0\}$, and $\{x_2=1\}$, and Neumann boundary conditions along $\{x_1=10\}$. Linearization about this base solution results in the nonsymmetric eigenvalue problem

$$\begin{aligned} -\nu\Delta v_1 + x_2\partial_1 v_1 + v_2 &= \lambda v_1, \\ -\nu\Delta v_2 + x_2\partial_1 v_2 &= \lambda v_2, \end{aligned} \tag{7.6}$$

for $v=\{v_1,v_2\}$, with boundary condition $v_{|\partial\Omega}=0$. All of the resulting eigenvalues have positive real part. Due to the coupling of the second component v_2 into the first equation, the most critical eigenvalue λ^{crit} is expected to have defect $\alpha>0$. This property persists under discretization because of the particular structure of the problem. We introduce an additional coupling term $h^2 v_1$ in the second equation which makes the discrete eigenvalue λ_h^{crit} split into two simple (real) eigenvalues λ_h^1 and λ_h^2. As suggested by the *a priori* error analysis, we take $\lambda_h := \frac{1}{2}\{\lambda_h^1+\lambda_h^2\}$ as our primary approximation to the limit eigenvalue λ. Table 1 presents the corresponding results obtained on a sequence of uniformly refined meshes for the parameter value $\nu=10^{-2}$. We observe the reduced order $\mathcal{O}(h)$ for the error $|\lambda_h^1-\lambda|$ and the optimal order $\mathcal{O}(h^2)$ for $|\lambda_h-\lambda|$. The same order is obtained for the error estimator $\eta^{\lambda}_{\text{weight}}$. Furthermore, as predicted, the dual eigenfunctions v_h^*, normalized by $(v_h,v_h^*)=1$ have norms that blow up with order $\mathcal{O}(h^{-1})$.

Table 1. Computation of the most critical eigenvalue $\lambda^{\text{crit}}=2.1228\ldots$ of the Burgers equation on uniform meshes ($\nu=10^{-2}$).

h	N	$\lvert\lambda_h^1-\lambda\rvert$	$\lvert\lambda_h-\lambda\rvert$	η^{λ}_{ω}	$\lVert v_h^*\rVert$
2^{-3}	162	$3.24e{-2}$	$7.07e{-3}$	$1.25e{-2}$	19.3
2^{-4}	578	$1.79e{-2}$	$1.80e{-3}$	$2.77e{-3}$	38.8
2^{-5}	2178	$9.42e{-3}$	$4.55e{-4}$	$6.54e{-4}$	77.9
2^{-6}	8450	$4.82e{-3}$	$1.16e{-4}$	$1.39e{-4}$	155.9
2^{-7}	33282	$2.43e{-3}$	$3.11e{-5}$	$3.47e{-5}$	311.9
2^{-8}	132098	$1.21e{-3}$	$8.02e{-6}$	$8.58e{-6}$	623.9
order		$\mathcal{O}(h)$	$\mathcal{O}(h^2)$	$\mathcal{O}(h^2)$	$\mathcal{O}(h^{-1})$

7.3. Test case 3: Stability of channel flow around a cylinder

We consider laminar flow around the cross section of a cylinder in a two-dimensional channel (with slightly displaced vertical position) as shown in Figure 5. The volume force is $f = 0$. This is a standard benchmark problem for which reference solutions are available (Schäfer & Turek [32]). At the boundary $\partial\Omega$, the usual non-slip condition $v|_{\Gamma_{\text{rigid}}} = 0$ is imposed together with parabolic inflow and free-stream outflow conditions $v|_{\Gamma_{\text{in}}} = v^{\text{in}}$ and $\nu\partial_n v - np|_{\Gamma_{\text{out}}} = 0$, respectively. A quantity of physical interest is, for example, the drag coefficient defined by

$$J_{\text{drag}} := \frac{2}{\bar{v}^2 D}\int_S n^T \sigma(v,p)e_x \,\mathrm{ds},$$

where S is the surface of the cylinder, D its diameter, $\bar{v}$ the reference velocity, and $\sigma(v,p) = \frac{1}{2}\nu(\nabla v + \nabla v^T) + pI$ the stress force acting on S.

We consider an optimization problem related to the above boundary value problem, namely the minimization of the 'cost-functional' $J(u) := J_{\text{drag}}$ for $u \in V + \{v^{\text{in}}, 0\}$, for 'boundary control' realized by imposing pressure mean values at two openings above and below the obstacle, forming the control boundary $\Gamma_Q = \Gamma_1 \cup \Gamma_2$. The corresponding state equation is

$$\begin{aligned} &-\nu\Delta v + v\cdot\nabla v + \nabla p = 0, \quad \nabla\cdot v = 0 \quad \text{in } \Omega, \\ &\hat{v}|_{\Gamma_{\text{rigid}}} = 0,\ \hat{v}|_{\Gamma_{\text{in}}} = \hat{v}^{in},\ \nu\partial_n\hat{v} - \hat{p}n|_{\Gamma_{\text{out}}} = 0,\ \nu\partial_n\hat{v} - \hat{p}n|_{\Gamma_Q} = q. \end{aligned} \tag{7.7}$$

Figure 5. Configuration of the drag minimization problem

Using the notation from above, the corresponding variational form reads

$$a(u;\varphi) + b(q,\varphi) = 0 \quad \forall\varphi \in V, \tag{7.8}$$

with the bilinear 'control form' $b(q,\varphi) := -(q, n\cdot\varphi^v)_{\Gamma_Q}$. The control q is chosen constant at the two components of the control boundary $\Gamma_Q = \Gamma_1 \cup \Gamma_2$ and therefore spans a two-dimensional control space $Q = \mathbb{R}^2$. Hence, the solution space for the optimal control problem is $\hat{V} \times \mathbb{R}^2$. After discretization the state equation becomes

$$a_h(u_h;\varphi_h) + b(q_h,\varphi_h) = 0 \quad \forall\varphi_h \in V_h, \tag{7.9}$$

with the notation from above. Since solving the state equation several times with good accuracy is expensive, the use of an economical discretization is indispensable. This leads us to the question of what degree of 'admissibility' of the approximate state u_h is needed for the optimization process. Our approach is based on the concept that the discretization of the state equation should be adapted according to the evaluation of the cost-functional $J(\cdot)$. The resulting 'optimal' controls q_h^{opt} and flow states u_h^{opt} usually satisfy the state equation (Navier–Stokes equations) only in a rather weak sense. This may raise concern about the physical relevance of this solution. However, the purpose of the 'model reduction' is only to minimize the costs of the optimization process. Once we have obtained a good optimal control q_h^{opt}, a more 'admissible' flow state $\tilde{u}_h^{\text{opt}} \in V_{h'}$ may be generated by approximating the state equation with optimal q_h^{opt} using an enhanced trial space $V_{h'}$:

$$a_h(\tilde{u}_h^{\text{opt}}; \varphi_h) + b(q_h^{\text{opt}}, \varphi_h) = 0 \quad \forall \varphi_h \in V_{h'}. \tag{7.10}$$

Usually, this 'post-processing' step is much cheaper compared to carrying out the whole optimization process in $V_{h'}$.

From Becker [3] and Becker et al. [5], we recall the following results for the above optimization problem. The data is chosen such that the Reynolds number is $\mathbf{Re} := \bar{v}^2 D/\nu = 40$ for the uncontrolled flow. Table 2 contains the values of the drag coefficient on optimized meshes as shown in Figure 6 with results obtained on globally refined meshes. It is clear from these numbers that a significant reduction in the dimension of the discrete model is possible by using appropriately adapted meshes. Figure 6 shows streamline plots of the uncontrolled ($q = 0$) and the controlled ($q = q^{\text{opt}}$) solution and a corresponding mesh adapted for the optimization process.

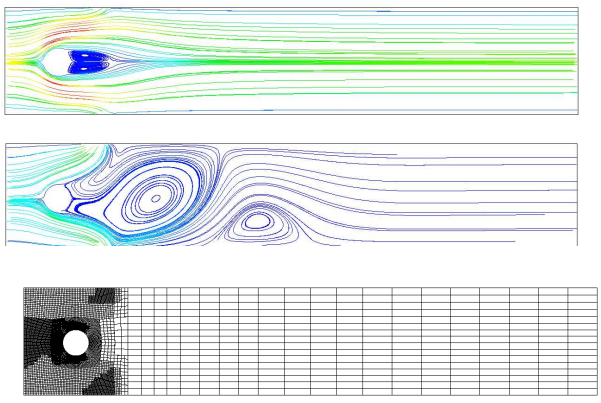

Figure 6. Velocity of the uncontrolled flow (top), the controlled flow (middle) and the corresponding adapted mesh (bottom).

Table 2. Uniform refinement (left) versus adaptive refinement (right).

Uniform refinement		Adaptive refinement	
N	J_{drag}	N	J_{drag}
10512	3.31321	1572	3.28625
41504	3.21096	4264	3.16723
164928	3.11800	**11146**	3.11972

The locally refined mesh produced by the adaptive algorithm seems to contradict intuition since the recirculation behind the cylinder is not so well resolved. However, due to the particular structure of the optimal velocity field (most of the flow leaves the domain at the control boundary), it might be clear that this recirculation does not significantly influence the cost functional. Instead, a strong local refinement near the cylinder, where the cost functional is evaluated, as well as near the control boundary is produced. However, the structure of the flow field of the drag-minimal solution indicates that this stationary flow may not be (dynamically) stable and hence not physical. To test this, we consider the corresponding stability eigenvalue problem

$$a'(u_h^{\text{opt}}; u_h, \varphi_h) = \lambda_h \, m(u_h, \varphi_h) \quad \forall \varphi_h \in V_h. \tag{7.11}$$

The state equation (7.8) has been solved for several values of the control q with mesh adaptation on the basis of the error estimator $\eta_h^{\lambda}(\hat{u}_h, \hat{u}_h^*, u_h, u_h^*, \lambda_h)$. Figure 7 displays the real and imaginary parts of the critical eigenvalue λ^{crit} as function of the control mean pressure q. The 'optimal' control pressure for the stationary model is around $q = 0.5$ and the corresponding flow state turns out to be unstable, Heuveline & Rannacher [18].

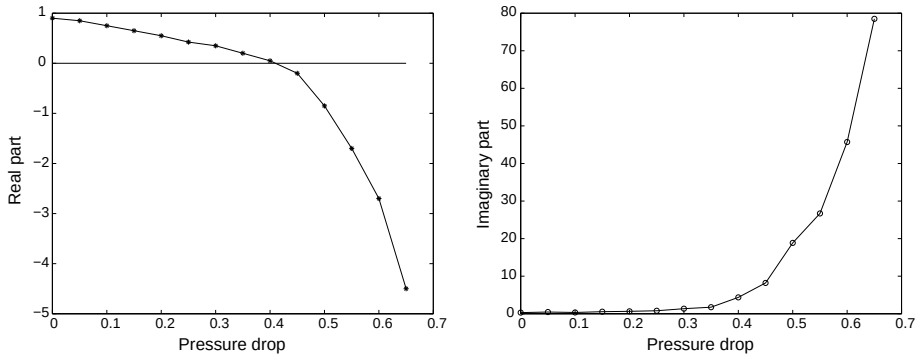

Figure 7. Real and imaginary parts of the critical eigenvalue as function of the control pressure.

In Figure 8 we show the size of the two components in the error estimator $\eta_h^\lambda(\hat{u}_h, \hat{u}_h^*, u_h, u_h^*, \lambda_h)$, measuring the error in the approximation of the base solution and the error in the eigenvalue approximation. On coarse meshes the error in approximating the base solution, i.e., the coefficient in the stability eigenvalue problem, dominates, while under further mesh refinement this error becomes smaller than that of the eigenvalue approximation. In particular, we see that on the finest mesh shown in Figure 9, which is adapted in accordance to the drag minimization process, we can reliably predict the instability of the corresponding flow state, Heuveline & Rannacher [18].

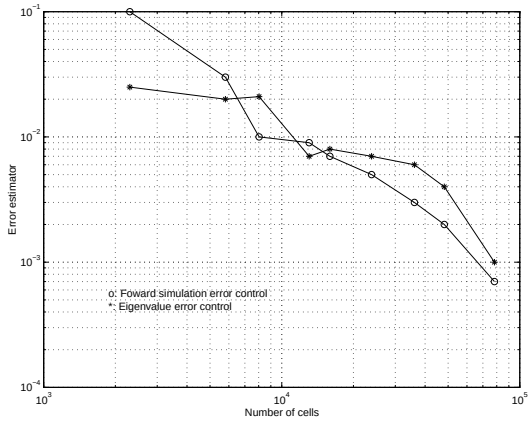

Figure 8. The size of the two components of the error estimator $\eta_h^\lambda(\hat{u}_h, \hat{u}_h^*, u_h, u_h^*, \lambda_h)$, i.e. the errors in the base solution and the eigenvalue approximation.

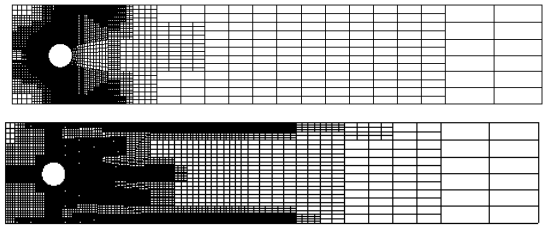

Figure 9. Meshes obtained by the error estimators for the drag minimization (top) and the eigenvalue computation (bottom).

8. CONCLUSION

We have derived *a posteriori* error estimates for the Galerkin finite element approximation of eigenvalue problems associated to the linearized Navier-Stokes equations. The argument is based on results of an abstract *a posteriori* theory for Galerkin discretization of general nonlinear problems that uses concepts from optimal control theory. By computational tests the resulting estimates have been demonstrated to be asymptotically reliable and efficient. In this report, we have primarily considered the case of well-conditioned nondeficient eigenvalues and also that with deficiency $\alpha > 0$. The aspect of nontrivial critical pseudo-spectra, only briefly addressed here, is analyzed in more detail in Heuveline & Rannacher [18].

REFERENCES

1. I. Babuska and J. E. Osborn. *Eigenvalue problems*. Handbook of Numerical Analysis, Vol. 2, Chapter Finite Element Methods, Part 1 (P.G. Ciarlet and J.L. Lions, eds), pp. 641–792, Elsevier, 1991.
2. W. Bangerth and R. Rannacher. *Adaptive Finite Element Methods for Differential Equations*. Birkhäuser: Basel-Boston-Berlin, 2003.
3. R. Becker. *Mesh adaptation for stationary flow control*. J. Math. Fluid Mech. 3, 317–341 (2001).
4. R. Becker. *Weighted error estimators for finite element approximations of the incompressible Navier-Stokes equations*. East-West J. Numer. Math. 8, 257–274 (2000).
5. R. Becker, V. Heuveline and R. Rannacher. *An optimal control approach to adaptivity in computational fluid mechanics*, Int. J. Numer. Meth. Fluids. 40, 105-120 (2002).
6. R. Becker and R. Rannacher. *A feed-back approach to error control in finite element methods: basic analysis and examples*. East-West J. Numer. Math. 4, 237–264 (1996).
7. R. Becker and R. Rannacher. *An optimal control approach to error control and mesh adaptation in finite element methods*. Acta Numerica 2001 (A. Iserles, ed.), Cambridge University Press, to appear, 2001.
8. M. Braack and R. Rannacher. *Adaptive finite element methods for low-Mach-number flows with chemical reactions*, Lecture Series 1999-03, 30th Computational Fluid Dynamics (H. Deconinck, ed.), von Karman Institute for Fluid Dynamics, Belgium, 1999.
9. J.H. Bramble and J.E. Osborn. *Rate of convergence estimates for nonselfadjoint eigenvalue approximations*. Math. Comp. 27, 525–545 (1973).
10. P.G. Ciarlet. *Finite Element Methods for Elliptic Problems*. North-Holland, Amsterdam, 1978.
11. K. Eriksson, D. Estep, P. Hansbo, and C. Johnson. *Introduction to adaptive methods for differential equations*. Acta Numerica 1995 (A. Iserles, ed.), pp. 105–158, Cambridge University Press, 1995.
12. C. R. Doering and J. D. Gibbon. *Applied Analysis of the Navier-Stokes Equations*. Cambridge University Press, Cambridge, 1995.
13. P. G. Drazin and W. H. Reid. *Hydrodynamic Stability*. Cambridge University Press, Cambridge, 1982.
14. G. P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. 1: Linearized Steady problems, Vol. 2: Nonlinear Steady Problems*. Springer, Berlin-Heidelberg-New York, 1998.

15. V. Girault and P. A. Raviart. *Finite Element Methods for Navier-Stokes Equations*. Springer, Berlin-Heidelberg 1986.

16. V. Heuveline and C. Bertsch. *On multigrid methods for the eigenvalue computation of nonselfadjoint elliptic operators*. East-West J. Numer. Math. 8, 275–297 (2000).

17. V. Heuveline and R. Rannacher. *A posteriori error control for finite element approximations of elliptic eigenvalue problems*. J. Comput. Math. Appl. 15, 107–138 (2001).

18. V. Heuveline and R. Rannacher. *Solution of stability eigenvalue problems by adaptive finite elements with application in hydrodynamic stability theory*. Institute of Applied Mathematics, University of Heidelberg, Preprint, Mai 2006.

19. V. Heuveline and F. Schieweck. *On the inf-sup condition for mixed hp-FEM on meshes with hanging nodes*. IWR and Institute of Applied Mathematics, University of Heidelberg, IWR-Preprint 2004-25, 2004.

20. T. J. R. Hughes and A. N. Brooks. *Streamline upwind/Petrov Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equation*. Comp. Math. Appl. Mech. Engrg 32, 199–259 (1982).

21. T. J. R. Hughes, L. P. Franca, and M. Balestra. *A new finite element formulation for computational fluid dynamics: V. Circumvent the Babuska-Brezzi condition: A stable Petrov-Galerkin formulation for the Stokes problem accomodating equal order interpolation*. Comp. Meth. Appl. Mech. Engrg 59, 89–99 (1986).

22. C. Johnson and R. Rannacher. *On error control in CFD*, Proc. Int. Workshop Num. Meth. Navier-Stokes Equations (F.-K. Hebeker, R. Rannacher, G. Wittum, eds), pp. 133–144, Notes Num. Fluid Mech., Vieweg, 1994.

23. C. Johnson, R. Rannacher and M. Boman. *Numerics and hydrodynamic stability: Towards error control in CFD*, SIAM J. Numer. Anal. 32, 1058–1079 (1995).

24. C. Johnson, R. Rannacher and M. Boman. *On transition to turbulence and error control in CFD*, Preprint 95-06, SFB 359, University of Heidelberg,1995.

25. W. Kolata. *Approximation in variationally posed eigenvalue problems*. Numer. Math., 29, 159-171 (9789.

26. T. Kato. *Perturbation Theory for linear Operators*. Springer, Berlin-Heidelberg-New York, 1966.

27. M. Landahl. *A note on an algebraic instability of viscous parallel shear flows*, J. Fluid Mech. 98, 243 (1980).

28. M.G. Larson. *A posteriori and a priori error analysis for finite element approximations of self-adjoint elliptic eigenvalue problems*. SIAM J. Numer. Anal. 38, 608–625 (2000).

29. C. Nystedt. *A priori and a posteriori error estimates and adaptive finite element methods for a model eigenvalue problem*. Technical Report N0 1995-05, Department of Mathematics, Chalmers University of Technology, 1995.

30. J. E. Osborn. *Spectral approximation for compact operators*. Math. Comp. 29, 712–725 (1975).

31. R. Rannacher. *Finite element methods for the incompressible Navier-Stokes equations*. Fundamental Directions in Mathematical Fluid Mechanics (G. P. Galdi, J. Heywood, R. Rannacher, eds), pp. 191–293, Birkhäuser, Basel-Boston-Berlin, 2000.

32. M. Schäfer and S. Turek. *The benchmark problem 'flow around a cylinder'*. Flow Simulation with High-Performance Computers (E. H. Hirschel, ed.), Notes Comput. Fluid Mech., Vieweg, Stuttgart, 1996.

33. L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll. *A new direction in hydrodynamical stability: Beyond eigenvalues*, Tech. Report CTC92TR115 12/92, Cornell Theory Center, Cornell University, 1992.

34. L. N. Trefethen. *Pseudospectra of linear operators*, In ICIAM 95, Proc. Third Int. Congr. on Industrial and Applied Mathematics (K. Kirchgässner, G. Mahrenholtz, and R. Mennicken, eds.), pp. 401–434, Akademie Verlag, Berlin, 1996.

35. R. Verfürth. *A Review of a posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*. John Wiley/Teubner, New York-Stuttgart, 1996.